Firm Dynamics and Labor Flows via Heterogeneous Agents

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FIRM DYNAMICS AND LABOR FLOWS VIA HETEROGENEOUS AGENTS

ABSTRACT

A model is described in which large numbers of simple agents organize into groups that empirically resemble U.S. firms. The agents work in team production environments, regularly adjust their work effort, and periodically seek better jobs or start new teams when it is in their self-interest. Nash equilibria of the team formation game exist but are unstable. Dynamics are studied using agent computing at full-scale with the U.S. private sector (120 million agents). Stationary distributions of firm sizes, ages, growth rates, wages, job tenure and so on arise at the aggregate level despite perpetual adaptation at the agent level. Such agent adjustments occur for microeconomic reasons without the need for external shocks.

Keywords: firm formation, team production, bounded rationality, unstable Nash equilibria, job tenure distribution, firm size distribution, agent-based computational model, non-equilibrium microeconomics, economic complexity

JEL classification codes: C63, C73, D22, L11, L22

Human beings, viewed as behaving systems, are quite simple. The apparent complexity of our behavior over time is largely a reflection of the complexity of the environment in which we find ourselves. Herbert Simon (1996 [1969]: 53)

1 Introduction

While it is conventional, in a wide variety of economic models, to assume that the economy is in general equilibrium, there is, in fact, substantial dynamism in real economies. Consider the U.S. private sector: over the last decade the workforce has ranged from 115 to 120 million employees annually, with nearly 3 million workers changing employers *each month* on average (Davis, Faberman and Haltiwanger 2006). Over this same period there were, each year, 5.7-6.0 million firms with employees of which, on average, 100 thousand went out of business *monthly* while a comparable number started up (Fairlie 2012). Such high levels of turnover in the American economy—1 in 40 workers changing employers monthly, 1 in 60 firms terminating its operations—portrays a kind of *perpetual economic flux* in the U.S. How are we to interpret such persistent adjustments and reorganizations of productive activities? Conventionally, they are believed to represent the reallocation of human resources to more productive uses (Caves 1998). But do they generate actual productivity gains at the firm level? Do such fluxes partially result from previous changes, e.g., filling jobs previously opened? Do they cause new fluxes in the next period? Are they produced by *exogenous* shocks, whether aggregate or firm-specific (e.g., technological or productivity-related), or are they due to *endogenous* agent interactions and decisions? If we stipulate that the economy is in general equilibrium then there is no way to realize micro-dynamics except by the imposition of external shocks. Can microeconomic models *endogenously* produce the kinds of dynamics observed empirically when the incentives agents have to change jobs are fully represented?

The main result of the research described here is a microeconomic model capable of producing, *without* exogenous shocks, firm and labor dynamics of the size and type experienced by the U.S. economy prior to the recent financial crisis. In addition to the nearly 3 million people who change jobs in the U.S. each month, about half as many, some 1.5 million workers, separate from their employers monthly without new jobs, becoming unemployed, while a comparable number move off unemployment into new jobs; another 1.5 million people either leave the workforce for a spell or else begin a job after being out of the workforce. These flows sum to approximately 9 million labor market events per month at steady-state (Fallick and Fleischman 2004). Further, many of the vacancies created by such *inter-firm* flows are filled by *intra-firm* job changes, about which there are fewer data. All told, perhaps 12 million distinct job change events occur each *month* in the U.S., involving nearly 10% of the 120 million people in the private sector. Clearly, over the course of a year there is enormous turnover in the matching of people to jobs in the U.S. While conventional explanations for these large labor flows exist (e.g., Krusell, Mukoyama, Rogerson and Sahin 2011), here I provide a microeconomic explanation without the need for aggregate shocks.

I also reproduce a variety of cross-sectional properties of U.S. businesses. Over the past decade there have appeared increasing amounts of micro-data on U.S. firms, including administratively *comprehensive* (tax record-based) data on firm sizes, ages, growth rates, labor productivity, job tenure, and wages. Extant theories place few restrictions on these data.[1] Lucas (1978) derives Pareto-distributed firm sizes from a Pareto distribution of managerial talent. Luttmer (2007, 2010) obtains Zipf-distributed firm sizes and exponential firm ages (2011) in a variety of general equilibrium settings, driven by exogenous shocks. Rossi-Hansberg and Wright (2007) study establishment growth and exit rates arising in general equilibrium due to industry-specific productivity shocks. Elsby and Michaels (2013) and Arkolakis (2013) simulate heterogeneous firm growth rates due to productivity shocks. However, there are *many* more data on firm dynamics and labor flows to be explained. Here I develop a model that reproduces more

[1] A generation ago Simon noted the inability of the neoclassical theory of the firm to explain the empirical size distribution (Ijiri and Simon 1977: 7-11, 138-140, Simon 1997). Transaction cost (e.g., Williamson 1985) and game theoretic explanations of the firm (e.g., Hart 1995, Zame 2007) make few empirical claims. Sutton (1998) bounds the extent of intra-industry concentration, constraining the shape of size distributions.

than three dozen features of the empirical data *without* recourse to exogenous shocks—such shocks are not necessary in a model with worker-level dynamics.

The model draws together threads from various theoretical literatures. It is written at the level of individual agents and incentive problems of the type studied in the principal-agent literature manifest themselves. The agents work in perpetually novel environments, so contracts are incomplete and transaction costs are implicit. Each firm is a coalition of agents making the theory of coalition formation relevant (Ray 2007). Agent decisions generate firm growth and decline in the spirit of evolutionary economics (Nelson and Winter 1982).

Specifically, the model consists of a heterogeneous population of agents with preferences for income and leisure. Production takes place under increasing returns to scale, so agents who work together can produce more output per unit effort than by working alone. However, agents act non-cooperatively[2]: they select effort levels that improve their own welfare, and may migrate between firms or start-up new firms when it is advantageous to do so. Analytically, Nash equilibria within a firm can be unstable. Large firms are ultimately unstable because each agent's compensation is imperfectly related to its effort level, making free-riding possible. Highly productive agents eventually leave large firms and such firms eventually decline. All firms have finite lives. The dynamics of firms perpetually forming, growing and perishing are studied. It will be shown that *this non-equilibrium regime provides greater welfare than equilibrium.*

These dynamics mean it is analytically difficult to relate agent level behavior to the aggregate outcomes. Therefore, features that emerge at the firm population level are studied using agent-based computing (Holland and Miller 1991, Vriend 1995, Axtell 2000, Tesfatsion 2002). In agent computing individual software objects represent people and have behavioral rules governing their interactions. Agent models are 'spun' forward in time and regularities emerge from the interactions (e.g., Grimm*, et al.* 2005). The shorthand for this is that macro-structure "grows" from the bottom-up. No equations governing the aggregate

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[2] For a cooperative game theoretic view of firms see Ichiishi (1993).

level are specified. Nor do agents have either complete information or correct models for how the economy will unfold. Instead, they glean data inductively from the environment and from their social networks, through direct interactions, and make imperfect forecasts of economic opportunities. (Arthur 1994). The macroscopic properties of the model *emerge* from the agent interactions. This methodology facilitates modeling agent heterogeneity (Kirman 1992), non-equilibrium dynamics (Arthur 2002), local interactions (Kirman 1997), and bounded rationality (Arthur 1991, Kirman 1993).

2 Dynamics of Team Production

Consider a group of agents A, $|A| = n$, engaged in team production, each agent contributing some amount of effort, generating team output.[3] Specifically, agent i has endowment $\omega_i > 0$ and contributes effort level $e_{i\in A} \in [0, \omega_i]$, to the group. The total effort of the group is then $E \equiv \sum_{i \in A} e_i$. The group produces output, O, as a function of E, according to $O(E) = aE + bE^\beta$, $\beta > 1$, without capital as in Hopenhayn (1992).[4] For $b > 0$ there are increasing returns to effort.[5] Increasing returns in production means that agents working together can produce more than they can as individuals.[6] To see this, consider two agents having effort levels e_1 and e_2, with $\beta = 2$. As individuals they produce total output $O_1 + O_2 = a(e_1 + e_2) + b(e_1^2 + e_2^2)$, while working together they make $a(e_1 + e_2) + b(e_1 + e_2)^2$. Clearly this latter quantity is at least as large as the former since $(e_1 + e_2)^2 \geq e_1^2 + e_2^2$. Agents earn according to a compensation rule. For now consider agents sharing total output equally: at the end of each period all output is sold for unit price and each agent receives an O/N share of the total output.[7] Agents have Cobb-Douglas

[3] The model derives from Canning (1995), Huberman and Glance (1998) and Glance *et al.* (1997).
[4] While $O(E)$ relates inputs to outputs, like a standard production function, E is not the choice of a single decision-maker, since it results from the actions of autonomous agents. Thus, $O(E)$ cannot be made the subject of a math program, as in conventional production theory, yet does describe production possibilities.
[5] Increasing returns at the firm level goes back at least to Marshall (1920) and was the basis of theoretical controversies in the 1920s (Sraffa 1926, Young 1928). Recent work on increasing returns is reprinted in Arthur (1994) and Buchanan and Yoon (1994). Colander and Landreth (1999) give a history of the idea.
[6] There are many ways to motivate increasing returns, including ‘four hands problems’: two people working together are able to perform a task that neither could do alone, like carrying a piano up a flight of stairs.
[7] The model yields roughly constant total output, so in a competitive market the price of output would be nearly constant. Since there are no fixed costs, agent shares sum to total cost, which equals total revenue. The shares can be thought of as either uniform wages in pure competition or profit shares in a partnership.

preferences for income and leisure, parameterized by θ.[8] All time not spent working is spent in leisure, so agent i's utility can be written as a function of its effort, e_i, and the effort of other agents, $E_{\sim i} \equiv E - e_i$ as

$$U_i\left(e_i;\theta_i,\omega_i,E_{\sim i},n\right)=\left(\frac{O\left(e_i;E_{\sim i}\right)}{n}\right)^{\theta_i}\left(\omega_i-e_i\right)^{1-\theta_i}. \quad (1)$$

2.1 Equilibrium of the Team Production Game

Consider the individual efforts of agents to be unobservable. From team output, O, each agent i determines E and, from its contribution to production, e_i, can figure out $E_{\sim i}$. Agent i then selects effort $e_i^*\left(\theta_i,\omega_i,E_{\sim i},n\right)=\arg\max_{e_i} U_i\left(e_i\right)$. For $\beta = 2$, in symbols, $e_i^*(\theta_i, \omega_i, E_{\sim i}) =$

$$\max\left[0,\frac{-a-2b\left(E_{\sim i}-\theta_i\omega_i\right)+\sqrt{a^2+4b\theta_i^2\left(\omega_i+E_{\sim i}\right)\left[a+b\left(\omega_i+E_{\sim i}\right)\right]}}{2b\left(1+\theta_i\right)}\right]. \quad (2)$$

Note that e^* does not depend on n but does depend on $E_{\sim i}$—the effort put in by the other agents. To develop intuition for the general dependence of e_i^* on its parameters, figure 1 plots it for $a = b = 1$ and $\omega_i = 10$, as functions of $E_{\sim i}$ and θ_i.

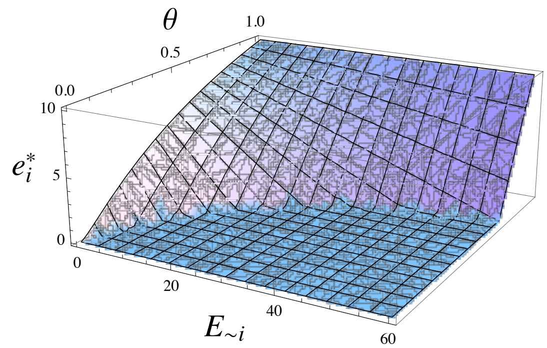

Figure 1*: Dependence of e_i^* on $E_{\sim i}$ and θ for $a = 1$, $b = 1$, $\omega_i = 10$*

Optimal effort decreases monotonically as 'other agent effort,' $E_{\sim i}$, increases. For each θ_i there exists some $E_{\sim i}$ beyond which it is rational for agent i to put in no effort. For constant returns, e_i^* decreases linearly with $E_{\sim i}$ with slope $\theta_i - 1$.

Singleton Firms

The $E_{\sim i} = 0$ solution of (2) corresponds to agents working alone in single

[8] Appendix A gives a more general model of preferences, yielding qualitatively identical results.

agent firms. For this case the expression for the optimal effort level is

$$e^*(\theta,\omega) = \frac{-a + 2b\theta\omega + \sqrt{a^2 + 4b\theta^2\omega(a + b\omega)}}{2b(1+\theta)} . \qquad (3)$$

For $\theta = 0$, $e^* = 0$ while for $\theta = 1$, $e^* = \omega$.

Nash Equilibrium

Equilibrium in a team corresponds to each agent working with effort e_i^* from (2), using $E_{\sim i}^*$ in place of $E_{\sim i}$ such that $E_{\sim i}^* = \sum_{j \neq i} e_j^*$. This leads to:

Proposition 1**:** Nash equilibrium exists and is unique (Watts 2002).

Proof: From the continuity of the *RHS*s of (2) and the convexity and compactness of the space of effort levels, a fixed point exists by the Brouwer theorem. Each fixed point is a Nash equilibrium, since once it is established no agent can make itself better off by working at some other effort level. □

Proposition 2: There exists a set of efforts that Pareto dominate Nash equilibrium (Hölmstrom 1982), a subset of which are Pareto optimal. These (a) involve larger effort levels than the Nash equilibrium, and (b) are not individually rational.

Proof: To see (a) note that $dU_i\left(e_i^*;\theta_i, E_{\sim i}^*, n\right) = \frac{\partial U_i}{\partial e_i} de_i + \frac{\partial U_i}{\partial E_{\sim i}} dE_{\sim i} > 0$, since the first term on the *RHS* vanishes at the Nash equilibrium and

$$\frac{\partial U_i}{\partial E_{\sim i}} = \frac{\theta_i\big(a + 2b(e_i + E_{\sim i})\big)(\omega_i - e_i)^{1-\theta_i}}{n^{\theta_i}\big[(e_i + E_{\sim i})\big(a + b(e_i + E_{\sim i})\big)\big]^{1-\theta_i}} > 0.$$

For (b), each agent's utility is monotone increasing on the interval [0, e_i^*), and monotone decreasing on (e_i^*, ω_i]. Therefore, $\partial U_i / \partial e_i < 0 \forall e_i > e_i^*, E_{\sim i} > E_{\sim i}^*$. □

The effort region that Pareto-dominates Nash equilibrium is the space where individuals who are part of the firm can achieve higher welfare than they do either working alone or at Nash equilibrium within the team.

Example 1*: Graphical depiction of the solution space 2 two identical agents*

Consider two agents with $\theta = 0.5$ and $\omega = 1$. Solving (2) for e^* with $E_{\sim i} = e^*$ and $a = b = 1$ yields $e^* = 0.4215$, corresponding to utility level 0.6704. Effort deviations by either agent alone are Pareto dominated by the Nash equilibrium, e.g., decreasing the first agent's effort to $e_1 = 0.4000$, with e_2 at the Nash level yields utility levels of 0.6700 and 0.6579, respectively. An effort increase to e_1 = 0.4400 with e_2 unchanged produces utility levels of 0.6701 and 0.6811, respectively, a loss for the first agent while the second gains. If both agents decrease their effort from the Nash level their utilities fall, while joint increases in effort are welfare-improving. There exist symmetric Pareto optimal efforts of 0.6080 and utility of 0.7267. However, efforts exceeding Nash levels are not individually rational—each agent gains by putting in less effort. Figure 2 plots iso-utility contours for these agents as a function of effort.

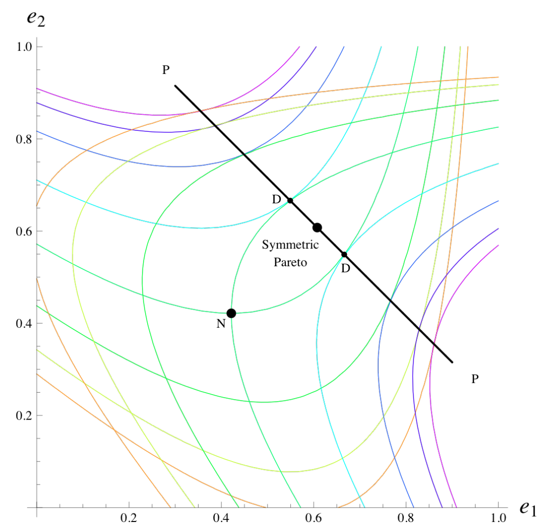

***Figure 2**: Effort level space for two agents with θ = 0.5 and a = b = ω = 1; colored lines are iso-utility contours, 'N' designates the Nash equilibrium, the heavy line from P-P are the Pareto optima, and the segment D-D represents the Pareto optima that dominate the Nash equilibrium*

The U-shaped lines are for the first agent, utility increasing upwards. The C-shaped curves refer to the second agent, utility larger to the right. Point 'N' is the Nash equilibrium. The 'core' shaped region extending above and to the right of 'N' is the set of efforts that Pareto-dominate Nash. The set of efforts from 'P' to 'P' are Pareto optimal, with the subset from ‘D’ to ‘D’ being Nash dominant.

For two agents with different θs the qualitative structure of the effort space shown in figure 1 is preserved, but the symmetry is lost. Increasing returns insures the existence of effort levels that Pareto-dominate the Nash equilibrium. For more than two agents the Nash and Pareto efforts continue to be distinct.

***Example 2**: Nash equilibrium in a team with free agent entry and exit*

Four agents having θs of {0.6, 0.7, 0.8, 0.9} work together with $a = b = \omega_i = 1$. Equilibrium, from (2), has agents working with efforts {0.15, 0.45, 0.68, 0.86}, respectively, producing 6.74 units of output. The corresponding utilities are {1.28, 1.20, 1.21, 1.32}. If these agents worked alone they would, by (3), put in efforts {0.68, 0.77, 0.85, 0.93}, generating outputs of {1.14, 1.36, 1.58, 1.80} and total output of 6.07. Their utilities would be {0.69, 0.80, 0.98, 1.30}. Working together they put in less effort and receive greater reward. This is the essence of team production. Now say a $\theta = 0.75$ agent joins the team. The four original members adjust their effort to {0.05, 0.39, 0.64, 0.84}—i.e., all work less—while total output rises to 8.41. Their utilities increase to {1.34, 1.24, 1.23, 1.33}. The new agent works with effort 0.52, receiving utility of 1.23, above its singleton utility of 0.80. If another agent having $\theta = 0.75$ joins the new equilibrium efforts of the original group members are {0.00, 0.33, 0.61, 0.83}, while the two newest agents contribute 0.48. The total output rises to 10.09 with utilities {1.37, 1.28, 1.26, 1.34} for the original agents and 1.26 for each of the twins. Overall, even though the new agent induces free riding, the net effect is a Pareto improvement. Next, an agent with $\theta = 0.55$ (or less) joins. Such an agent will free ride and not affect the effort or output levels, so efforts of the extant group members will not change. However, since output must be shared with one additional agent, all utilities fall. For the 4 originals these become {1.25, 1.15, 1.11, 1.17}. For the twins their utility falls to 1.12 and that of the $\theta = 0.9$ agent is now below what it can get working alone (1.17 vs 1.30). Since agents may exit the group freely, it is rational for this agent to do so, causing further adjustment: the three original agents work with efforts {0.10, 0.42, 0.66}, while the twins adds 0.55 and the newest agent free rides. Output is 7.52, yielding utility of {1.10, 0.99, 0.96} for the original three, 0.97 for the twins, and 1.13 for the free rider. Unfortunately for the group, the $\theta = 0.8$ agent now can do better by working alone—utility of 0.98 versus 0.96, inducing further adjustments: the original two work with efforts 0.21 and 0.49, respectively, the twins put in effort of 0.61, and the $\theta = 0.55$ agent rises out of free-riding to work at the 0.04 level; output drops to 5.80. The utilities of the originals are now 0.99 and 0.90, 0.88 for the twins, and 1.07 for the newest agent. Now the $\theta = 0.75$ agents are indifferent to staying or starting new singleton teams.

Homogeneous Teams

Consider a team composed of agents of the same type (identical θ and ω). In a homogeneous group each agent works with the same effort in equilibrium, determined from (2) by substituting (n-1) e_i^* for $E_{\sim i}$, and solving for e^*, yielding:

$$e^*(\theta,\omega,n) = \frac{2b\theta\omega n - a\big(\theta + n(1-\theta)\big) + \sqrt{4b\theta^2\omega n(a + b\omega n) + a^2\big(\theta + n(1-\theta)\big)^2}}{2bn\big(2\theta + n(1-\theta)\big)}. \quad (4)$$

These efforts are shown in figure 3a as a function of θ with $a = b = \omega = 1$ and various n. Figure 2b plots the utilities for $\theta \in \{0.5, 0.6, 0.7, 0.8, 0.9\}$ versus n.

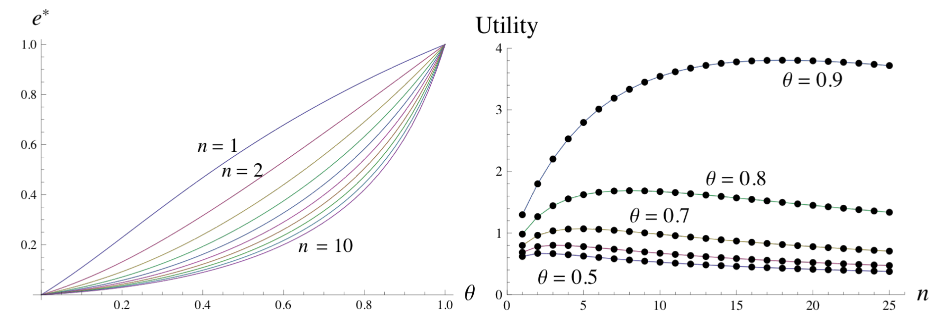

Figure 3*: Optimal effort (a) and utility (b) in homogeneous teams, vs. θ and n, with a = b = ω = 1*

Note that each curve in figure 3b is single-peaked, so there is an optimal team size for every θ. This size is shown in figure 4a for two values of ω.

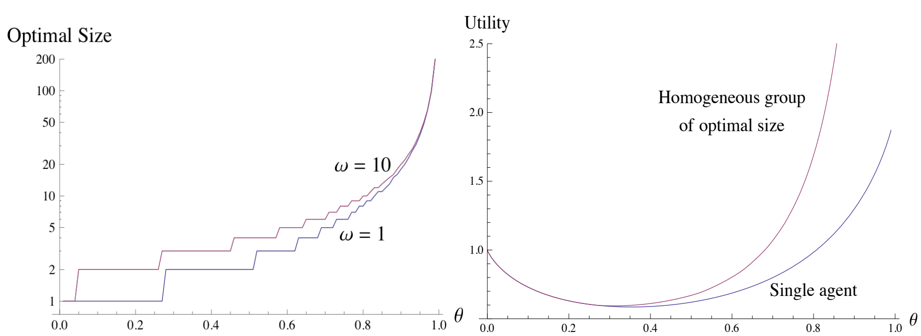

Figure 4*: Optimal size (a) and utility (b) in homogeneous teams versus θ; a = b = 1; ω = 1, 10*

Optimal team sizes rise quickly with θ (note log scale). Utilities in teams are shown in figure 4b. Gains from being in a team are greater for high θ agents.[9]

2.2 Stability of Nash Equilibrium, Dependence on Team Size

A unique Nash equilibrium always exists but for sufficiently large group size it is unstable. To see this, consider a team operating away from equilibrium, each agent adjusting its effort. As long as the adjustment functions are decreasing in $E_{\sim i}$ then one expects the Nash levels to obtain. Because aggregate effort is a linear combination of individual efforts, the adjustment dynamics can be conceived of in aggregate terms. In particular, the total effort level at time $t + 1$, $E(t+1)$, is a decreasing function of $E(t)$, as depicted notionally in figure 5 for a five agent firm, with the dependence of $E(t+1)$ on $E(t)$ shown as piecewise linear.

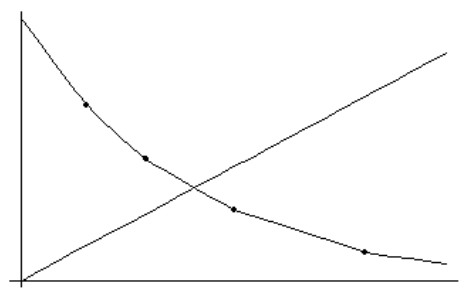

***Figure 5**: Phase space of effort level adjustment, n = 5*

The intersection of this function with the 45° line is the equilibrium total effort. However, if the slope at the intersection is less than –1, the equilibrium will be unstable. Thus, every team has a maximum stable size, dependent on agent θs.

Consider the n agent group in some state other than equilibrium at time t, with effort levels, $e(t) = (e_1(t), e_2(t), ..., e_n(t))$. At $t + 1$ let each agent adjust its effort using (2), a 'best reply' to the previous period's value of $E_{\sim i}$,[10]

$$e_i(t+1) = \max\left[0, \frac{-a - 2b\big(E_{\sim i}(t) - \theta_i\omega_i\big) + \sqrt{a^2 + 4b\theta_i^2\big(\omega_i + E_{\sim i}(t)\big)\big[a + b\big(\omega_i + E_{\sim i}(t)\big)\big]}}{2b\big(1+\theta_i\big)}\right].$$

9 For analytical characterization of an equal share (partnership) model with perfect exclusionary power see Farrell and Scotchmer (1988); an extension to heterogeneous skills is given by Sherstyuk (1998).

10 Effort adjustment functions that are decreasing in $E_{\sim i}$ and increasing in θ_i yield qualitatively similar results; see appendix A. While this is a dynamic strategic environment, agents make no attempt to deduce optimal multi-period strategies. Rather, at each period they myopically 'best respond'. This simple behavior is sufficient to produce very complex dynamics, suggesting sub-game perfection is implausible.

This results in an n-dimensional dynamical system, for which it can be shown:

Proposition 3: All teams are unstable for sufficiently large group size.

Proof: Start by assessing the eigenvalues of the Jacobian matrix:[11]

$$J_{ij} \equiv \frac{\partial e_i}{\partial e_j} = \frac{1}{1+\theta_i}\left\{\theta_i^2 \frac{a + 2b(\omega_i + E_{\sim i}^*)}{\sqrt{a^2 + 4b\theta_i^2(\omega_i + E_{\sim i}^*)[a + 2b(\omega_i + E_{\sim i}^*)]}} - 1\right\}, ()$$

with J_{ii} = 0. Since each $\theta_i \in$ [0, 1] it can be shown that $J_{ij} \in$ [-1, 0], and J_{ij} is monotone increasing with θ_i,. The *RHS* of () is independent of j, so each row of the Jacobian has the same value off the diagonal, i.e., $J_{ij} \equiv k_i$ for all $j \neq i$. Overall,

$$J = \begin{bmatrix} 0 & k_1 & \cdots & k_1 \\ k_2 & 0 & \cdots & k_2 \\ \vdots & & \ddots & \vdots \\ k_n & \cdots & k_n & 0 \end{bmatrix},$$

with each of the $k_i \leq 0$. Stability of equilibrium requires that this matrix's dominant eigenvalue, λ_0, have modulus strictly inside the unit circle. It will now be shown that this condition holds only for sufficiently small group sizes. Call ρ_i the row sum of the i^{th} row of J. It is well-known (Luenberger 1979: 194-195) that $\min_i \rho_i \leq \lambda_0 \leq \max_i \rho_i$. Since the rows of J are comprised of identical entries

$$(n-1)\min_i k_i \leq \lambda_0 \leq (n-1)\max_i k_i \,. \qquad ()$$

Consider the upper bound: when the largest $k_i < 0$ there is some value of n beyond which $\lambda_0 < -1$ and the solution is unstable. Furthermore, since large k_i corresponds to agents with high θ_i, it is these agents who determine group stability. From (A.2), compute the maximum stable group size, $N^{\max}$, by setting $\lambda_0 = -1$ and rearranging:

$$n^{\max} \leq \left\lfloor \frac{\max_i k_i - 1}{\max_i k_i} \right\rfloor, \qquad ()$$

where $\lfloor z \rfloor$ refers to the largest integer less than or equal to z. Groups larger than $n^{\max}$ will never be stable, that is, (A.3) is an upper bound on group size. ∎

[11] Technically, agents who put in no effort do not contribute to the dynamics, so the effective dimension of the system will be strictly less than n when such agents are present.

For any of b, $E_{\sim i}$ or $\omega_i \gg a$, such as when $a \sim 0$, $k_i \approx (\theta_i - 1)/(\theta_i + 1)$. Using this together with (A.3) we obtain an expression for n^{max} in terms of preferences

$$n^{\max} \le \left\lfloor \frac{2}{1 - \max_i \theta_i} \right\rfloor . \qquad ()$$

The agent with *highest* income preference thus determines the maximum stable group size. Other bounds on λ_0 can be obtained via column sums of J. Noting the i^{th} column sum by γ_i, we have $\min_i \gamma_i \le \lambda_0 \le \max_i \gamma_i$, which means that

$$\sum_{i=1}^{n} k_i - \min_i k_i \le \lambda_0 \le \sum_{i=1}^{n} k_i - \max_i k_i . \qquad ()$$

These bounds on λ_0 can be written in terms of the group size by substituting $n\,\bar{k}$ for the sums. Then an expression for n^{max} can be obtained by substituting $\lambda_0 = 1$ in the upper bound of (A.5) and solving for the maximum group size, yielding

$$n^{\max} \le \left\lfloor \frac{\max_i k_i - 1}{\bar{k}} \right\rfloor . \qquad ()$$

The bounds given by (A.3) and (A.6) are the same (tight) for homogeneous groups, since the denominators are identical in this case.

***Example 3**: Onset of Instability with Team Size*

Consider a homogeneous group of agents having $\theta = 0.7$, with $a = b = \omega = 1$. From (A.4) the maximum stable group size is 6. Here we investigate how instability arises as the group grows. For an agent working alone the optimal effort, from (3), is 0.770, utility is 0.799. Now imagine two agents working together. From (4) the Nash efforts are 0.646 and utility increases to 0.964. Each element of the Jacobian () is identical; call this k. For $n = 2$, $k = -0.188 = \lambda_0$. For $n = 3$ the utility is higher and $\lambda_0 = -0.368$. The same qualitative results hold for group sizes 4 and 5, with λ_0 approaching -1. At $n = 6$ efforts again decline and now each agent's utility is lower. Adding one more agent to the group ($n = 7$) causes λ_0 to fall to -1.082: the group is *unstable*—any perturbation of the Nash equilibrium creates dynamics that do not settle down. All of this is summarized in table 1.

n	e^*	$U(e^*)$	k	$\lambda_0 = (n-1)k$
1	0.770	0.799	not applicable	not applicable
2	0.646	0.964	-0.188	-0.188
3	0.558	1.036	-0.184	-0.368
4	0.492	1.065	-0.182	-0.547
5	0.441	1.069	-0.181	-0.726
6	0.399	1.061	-0.181	-0.904
7	0.364	1.045	-0.180	-1.082

***Table 1**: Onset of instability in a group having θ = 0.7; Nash eq. in groups larger than 6 are unstable*

Groups of greater size are also unstable in this sense. For lesser θ instability occurs at smaller sizes, while groups having higher θ can support larger numbers. Figure 6 shows the maximum stable firm size (in green) for all θ with $a = b = \omega = 1$, with the smallest size at which instability occurs (red). The lower (magenta) line is the optimal firm size (figure 4a), which is very near the stability boundary, sometimes in the unstable region.

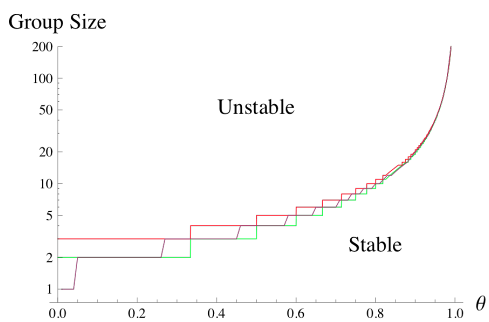

***Figure 6**: Unstable Nash equilibria in homogeneous groups having income preference θ*

This is reminiscent of the 'edge of chaos' literature, for systems poised at the boundary between order and disorder (2002).

Unstable Equilibria and Pattern Formation Far From Agent Level Equilibria

Unstable equilibria may be viewed as problematical if one assumes agent level equilibria are *necessary* for social regularity. Games in which optimal strategies are cycles have long been known (e.g., Shapley 1964, Shubik 1997). Solution concepts can be defined to include such possibilities (Gilboa and Matsui 1991). While agent level equilibria are *sufficient* for macro-regularity, they are not *necessary*. When agents are learning or in combinatorially rich environments, as they are here, fixed points are unlikely to be realized. Non-equilibrium models in economics include Papageorgiou and Smith (1983) and Krugman (1996).[12]

Real firms are inherently dynamic: workers leave, new ones arrive, everyone adjusts.[13] Indeed, there is vast turnover of jobs and firms, as already indicated. Of the largest 5000 U.S. firms in 1982, in excess of 65% of them no longer existed as independent entities by 1996 (Blair, Kruse and Blasi 2000)! 'Turbulence' well describes such volatility (Beesley and Hamilton 1984, Ericson and Pakes 1995).

3 From One Team to Six Million Firms

Consider a large population of agents in which many teams form simultaneously. If one or more of these teams becomes unstable some of the agents will look for employment in other teams, or perhaps they will form new teams if it makes them better off. What happens overall? Do lots of little teams form or a few big ones? Is a static equilibrium of agents in teams reached if we wait long enough? Are patterns produced in the population of teams that are recognizable vis-á-vis real firms? Here I show that such patterns *do* arise and can be made to closely resemble data on U.S. firms.

3.1 Realizing Realistic Firms

I study the formation of teams within a population using software agents. The set-up for the agent-based model is just like the analytical one. Total output of a firm consists of both constant and increasing returns. Preferences and endowments, θ and ω respectively, are heterogeneous across agents. When agent i acts it searches over $[0, \omega_i]$ for the effort maximizing its next period utility. Because many firms will arise in the computational model, it is necessary to specify how agents move between firms. Each agent has an exogenous social network, an Erdös-Renyi graph, consisting of ν_i other agents. It considers (a) staying in its current firm, (b) joining ν_i other firms—in essence an on-the-job search over its social network (Granovetter 1973, Montgomery 1991)—and (c) starting up a new firm. It chooses the option that yields greatest utility. Since agents evaluate only a small number of firms their information is very limited.

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[12] Non-equilibrium models are better known and well-established in other sciences, e.g., in mathematical biology the instabilities of certain PDE systems are the basis for pattern formation (Murray 1993).

[13] Arguments against firm equilibrium include Kaldor (1972, 1985), Moss (1981) and Lazonick (1991).

We utilize 120 million agents, roughly the size of the U.S. private sector. Specifically, about 5 million agents are activated each period, corresponding to one calendar month, in rough accord with job search frequency (Fallick and Fleischman 2001) and closely approximating the distribution of job tenure. The 'base case' parameterization of the model in table 2 was developed by seeking good fits to the many empirical data described in the next three subsections.[14].

Model Attribute	Value
number of agents	120,000,000
constant returns coefficient, a	*uniform* on [0, 1/2]
increasing returns coefficient, b	*uniform* on [3/4, 5/4]
increasing returns exponent, β	*uniform* on [3/2, 2]
distribution of preferences, θ	*uniform* on [0, 1]
endowments, ω	1
compensation rule	equal shares
number of neighbors, ν	*uniform* on [2,6]
activation regime	uniform (all agents active each period)
probability of agent activation/period	4% of total agents (4,800,000)
time calibration: one model period	one month of calendar time
initial condition	all agents in singleton firms

***Table 2**: 'Base case' configuration of the computational model*

Execution of the model can be summarized in pseudo-code:

```
• INSTANTIATE and INITIALIZE time, agent, firm, and data objects;
• REPEAT:
    o FOR each agent, activate it with 4% probability:
        ▪ Compute e* and U(e*) in current firm;
        ▪ Compute e* and U(e*) for starting up a new firm;
        ▪ FOR each firm in the agent's social network:
            • Compute e* and U(e*);
        ▪ IF current firm is not best choice THEN leave:
            • IF start-up firm is best THEN form start-up;
            • IF another firm is best THEN join other firm;
    o FOR each firm:
        ▪ Sum agent inputs and then do production;
        ▪ Distribute output;
    o COLLECT monthly and annual statistics;
    o INCREMENT time and reset periodic statistics;
```

Each worker is represented as an agent in this model, and both agents and firms are software objects. It is important to emphasize that this is *not* a numerical model: there are no (explicit) equations governing the aggregate level; the only equations present are for agent decisions. "Solving" an agent model means

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[14] For model attributes with random values, each agent or firm is given a realization when it is instantiated.

marching it forward in time to see what patterns emerge (cf. Axtell 2000).

3.2 Aggregate Dynamics

Initially, agents work alone. As each is activated it discovers it can do better working with another agent to jointly produce output. Over time some teams expand as certain agents find it welfare-improving to join them, while other teams contract as their agents discover better opportunities elsewhere. New firms are started-up by agents who lack better opportunities. Overall, once an initial transient passes an approximately stationary macrostate emerges.[15] In this macro steady-state agents continue to adjust their efforts and change jobs, causing firms to evolve, and so there is no equilibrium at the agent level.

Number of Firms and Average Firm Size

The number of firms varies over time, due both to entry—agents leaving extant firms for start-ups—and the demise of failing firms. In the U.S. about 6 million firms have employees. Figure 7 shows the number of firms (blue) in the steady-state over 25 years (300 months), in good agreement with the data.

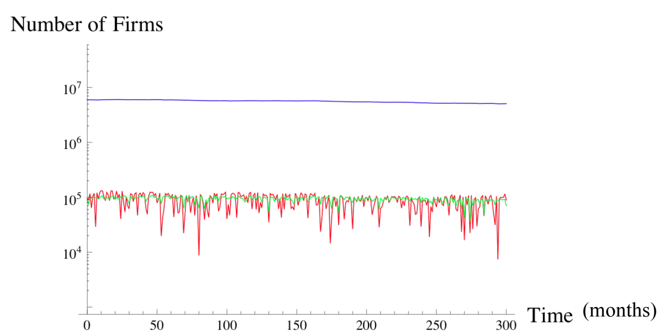

***Figure** 7: Typical time series for the total number of firms (blue), new firms (green), and exiting firms (red) over 25 years (300 months); note higher volatility in exits.*

There are ~100K startups with employees in the U.S. monthly (Fairlie 2012), quite close to the number produced by the model as shown in figure 6 (green). Exits are shown in red. The model predicts higher variability in firm exit than entry. Mean firm size in the U.S. is about 20 workers/firm (Axtell 2001). Since there are 120 million agents in the model and the number of firms that emerges is

[15] Movies are available at `css.gmu.edu/~axtell/Rob/Research/Pages/Firms.html#6`.

approximately 6 million, mean firm size, as shown in figure 8, is very close to 20.

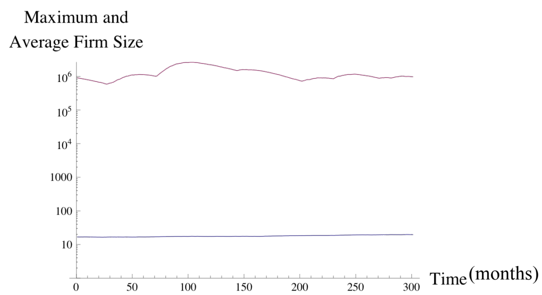

***Figure 8**: Typical time series for average firm size (blue) and maximum firm size (magenta)*

Also shown in figure 7 is the size of largest firm (red), which fluctuates around a million. The largest firm in the U.S. (Wal-Mart) employs about 1.3 million today.

Typical Effort and Utility Levels

Agents who work together improve upon their singleton utility levels with reduced effort, as shown in figure 9. This is the *raison d'être* of firms.

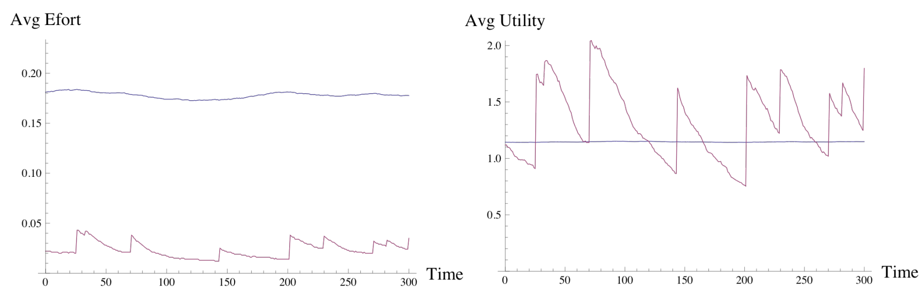

***Figure 9**: Typical time series for (a) average effort level in the population (blue) and in the largest firm (magenta), (b) average utility (blue) and in the largest firm (magenta)*

While efforts in large firms fluctuate, average effort overall is quite stable (figure 8a). Much of the dynamism in the 'large firm' time series is due to the identity of the largest firm changing. Figure 8b shows the average agent utility (blue) is usually less than that in the largest firm (red). Occasionally utility in large firms falls below average, signaling that the large firm is in decline.

Labor Flows

In the U.S. economy people change jobs with, what is to some, "astonishingly high" frequency (Hall 1999: 1151). Job-to-job switching (aka employer-to-employer flow) represents 30-40% of labor turnover, substantially higher than unemployment flows (Fallick and Fleischman 2001, Faberman and Nagypál

2008, Nagypál 2008, Davis, Faberman and Haltiwanger 2012). Moving between jobs is intrinsic to this model. In figure 10 the level of monthly job changing at steady-state is shown (blue)—just over 3 million/month—along with measures of jobs created (red) and jobs destroyed (green). Job creation occurs in firms with net monthly hiring, while job destruction means firms lose workers (net). Job destruction is about 4x more volatile than job creation, comparable to U.S. data (Davis, Haltiwanger and Schuh 1996).

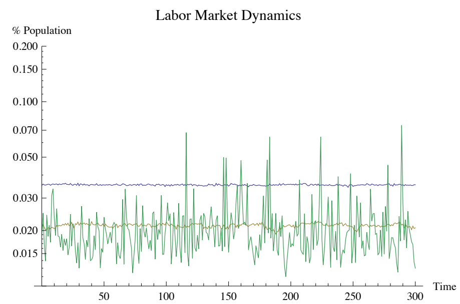

***Figure 10**: Typical monthly job-to-job changes (blue), job creation (yellow) and destruction (green)*

Overall, figures 7-10 develop intuition about typical dynamics of firm formation, growth and dissolution. They are a 'longitudinal' picture of typical micro-dynamics of agents and firms. We now look at cross-sectional properties.

3.3 Firms in Cross-Section: Sizes, Ages and Growth Rates

Watching firms form, grow, and die in the model movies (see footnote 14), one readily sees the coexistence of big firms, medium-sized ones, and small ones.

Firm Sizes (by Employees and Output)

At any instant there exists a distribution of firm sizes in the model. At steady-state, sizes are skew, with a few big firms and larger numbers of progressively smaller ones. Typical model output is shown in figure 11 for firm size measured by employees and output. The modal firm size is 1 employee with the median between 3 and 4, in agreement with the data on U.S. firms. Firm sizes, *S*, are approximately Pareto distributed, the complementary *CDF* of which, $\bar{F}_S(s)$ is

$$Pr[S \geq s] \equiv \bar{F}_S(s;\ \alpha, s_0) = \left(\frac{s_0}{s}\right)^{\alpha}, s \geq s_0, \alpha \geq 0$$

where s_0 is the minimum size, unity for size measured by employees. The U.S. data are well fit by $\alpha \approx$ -1.06 (Axtell 2001), the line in figure 10a.

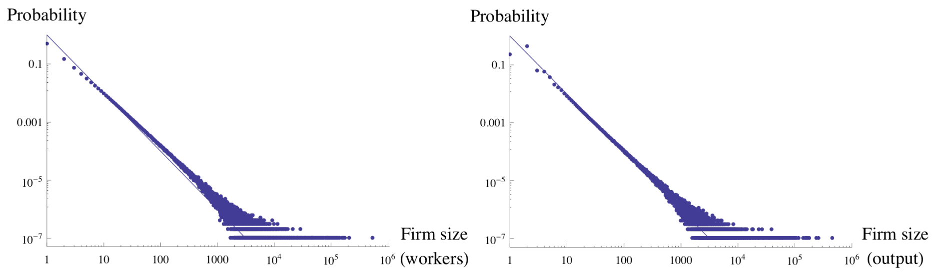

Figure 11*: Stationary firm size distributions (PMFs) by (a) employees and (b) output*

The Pareto is a power law, and for $\alpha = 1$ is known as Zipf's law. Note that the power law fits almost the *entire distribution* of firm sizes. A variety of explanations for power laws have been proposed.[16] Common to these is the idea that such systems are far from (static) equilibrium at the microscopic (agent) level. Our model is non-equilibrium with agents regularly changing jobs.

Labor Productivity

Firm output per employee is labor productivity. Figure 12 plots average firm output as a function of firm size. Fitting a line by several methods indicates that $ln(O)$ scales linearly with $ln(S)$ with slope very nearly 1.

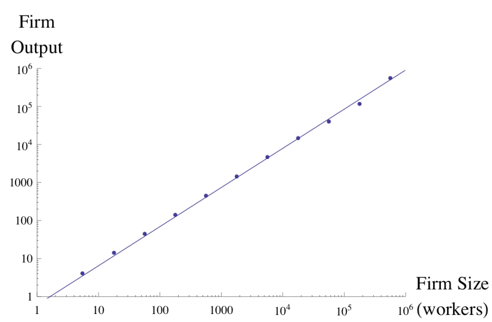

Figure 12*: Constant returns at the aggregate level despite increasing returns at the micro-level*

This represents essentially constant returns to scale, also a feature of U.S. output data; see Basu and Fernald (1997). That nearly *constant returns* occur at the *aggregate* level despite *increasing returns* at the *micro*-level suggests the difficulties of making inferences across levels. An explanation of why this occurs is apparent. High productivity firms grow by adding agents who work less hard than incumbents, thus such firms are driven toward the average productivity. In

[16] Bak (1996: 62-64), Marsili and Zhang (1998), Gabaix, (1999), Reed (2001), and Saichev *et al*. (2010); for a review see (Mitzenmacher 2004).

essence, each agent who changes jobs 'arbitrages' returns across firms.[17]

It is well known that there is large heterogeneity in labor productivity across firms (e.g., Dosi 2007). Shown in figure 13a are data on all U.S. companies for three size classes: 1-99 employees (blue), 100-9,999 (red) and 10,000+ (green).

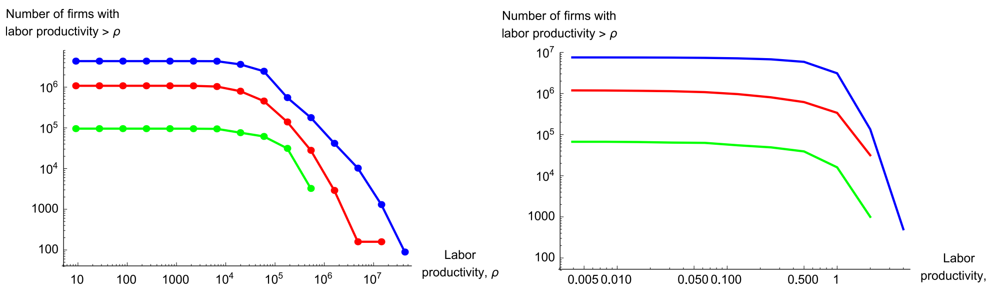

***Figure 13**: Labor productivity (a) U.S. data (Census) and (b) model output in arbitrary units*

Note the *log-log* coordinates, so the right tail is very nearly a power law with large slope. Souma *et al.* (2009) have studied the productivity of Japanese firms and find similar results. Figure 13b is model output for the same size classes.

Firm Ages, Survival Rates and Lifetimes

Using data from the BLS Business Employment Dynamics program, figure 14 gives the age distribution (*PMF*) of U.S. firms, in semi-log coordinates, with each colored line representing the distribution reported in a recent year.

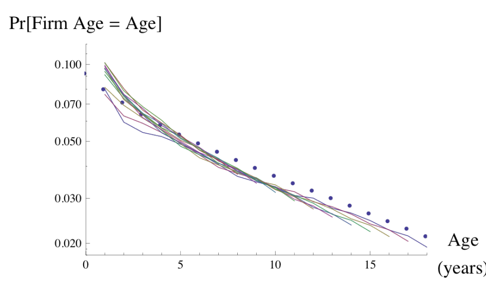

***Figure 14**: Firm age distributions (PMFs), U.S. data 2000-2011 (lines) and model output (points); source: BLS (www.bls.gov/bdm/us_age_naics_00_table5.txt) and author calculations*

Model output is overlaid on the raw data as points and agrees reasonably well. Average firm lifetime and standard deviation are 14-15 years here. The curvature in the data implies that firm ages are better fit by the Weibull distribution than the exponential (Coad 2010).

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[17] As output per worker represents wages in our model, there is little wage-size effect (Brown and Medoff 1989, Even and Macpherson 2012).

Data on U.S. firm ages is right censored so little systematic information is known about long-lived firms, except that they are rare (de Geus 1997). Further, the role of mergers and acquisitions (M&A) makes the lifetime of a firm ambiguous, as when a younger firm buys an older one. This model can be run for a long time and makes strong predictions about the distribution of firm ages, along with the closely related idea of firm lifetimes, as shown in figure 15.

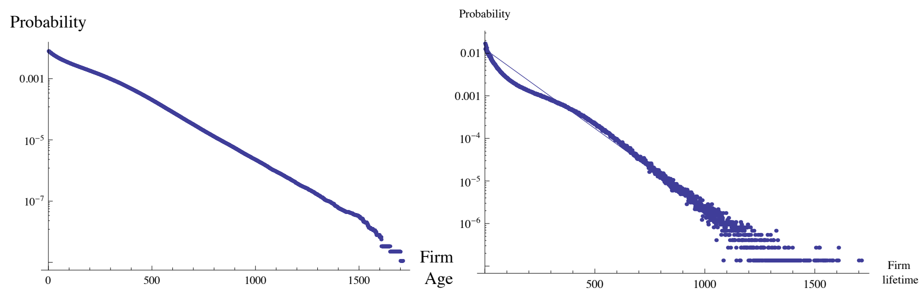

Figure 15*: Firm age distributions and lifetime distributions (PMFs) in the long run (months)*

If firm ages were exactly exponentially distributed then the survival probability would be constant, independent of age (Barlow and Proschan 1965). Curvature in figure 13 indicates that survival probability depends on age. Empirically, survival probability *increases* with age (Evans 1987, Hall 1987, Haltiwanger, Jarmin and Miranda 2011). This is shown in figure 16 for U.S. companies in recent years (lines) along with model output (points).

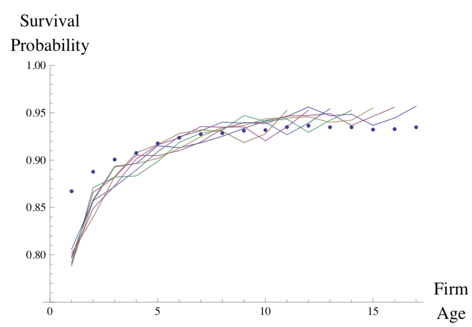

Figure 16*: Firm survival probability increases with firm age and size, U.S. data 1994-2000 (lines) and model (points); source: BLS and author calculations*

The model slightly over-predicts the survival probabilities of young firms.

Joint Distribution of Firms by Size and Age

The joint distribution of size and age is shown in figure 17, a normalized histogram in *log* probabilities.

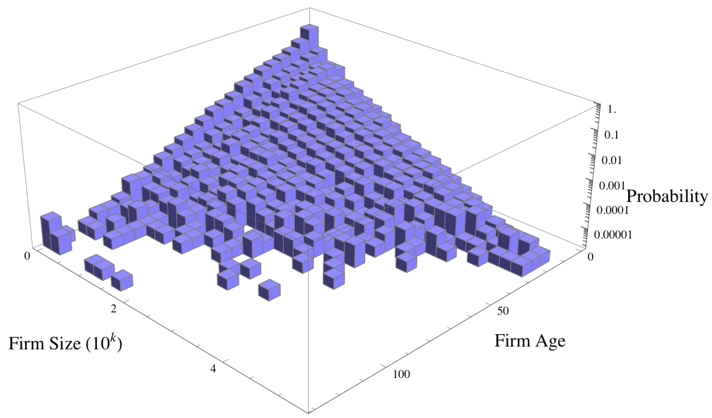

***Figure 17**: Histogram of the steady-state distribution of firms by log(S) and age in the model*

Note that log probabilities decline approximately linearly as a function of age and *log*(*S*). Many of the largest firms in the model are relatively young ones that grow rapidly, much like in the U.S. economy (e.g., Luttmer 2011, figure 1).

Firm Growth Rates

Call S_t a firm's size at time *t*. Its one period growth rate is $G \equiv S_{t+1}/S_t \in \boldsymbol{R}_+$.[18] In a population of firms consider *G* to be a stationary random variable. Gibrat's law of proportional growth (1931) implies that if all firms have the same *G* then $S_{t+1} = GS_t$ is lognormally distributed at any *t* while the mean and variance of *S* grows with time (Sutton 1997: 40), i.e., *S* is not stationary. Adding firm birth and death processes can lead to stationary firm size distributions (see de Wit (2005)).

Historically, determination of the overall structure of *G* was limited by the relatively small samples of firm data available (e.g., Hart and Prais 1956). Beginning with Stanley *et al*. (1996), who analyzed data on publicly-traded U.S. manufacturing firms (Compustat), there has emerged a consensus that $g \equiv ln(G) \in \boldsymbol{R}$ is well-fit by the Subbotin or exponential power distribution.[19] This

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[18] An alternative definition of *G* is $2(S_{t+1} - S_t)/(S_t + S_{t+1})$, making $G \in [-2, 2]$ (Davis, Haltiwanger and Schuh 1996). Although advantageous because it keeps exiting and entering firms in datasets for one additional period, it obscures differences in growth rate tails by artificially truncating them..

[19] Subsequent work includes European pharmaceuticals (Bottazzi, Dosi, Lippi, Pammolli and Riccaboni 2001) and Italian and French manufacturers (Bottazzi, Cefis, Dosi and Secchi 2007, Bottazzi, Coad, Jacoby and Secchi 2011). Bottazzi and Secchi (2006) give theoretical reasons why *g* should have $\eta \sim 1$, having to do with the central limit theorem for the number of summands geometrically distributed (Kotz, Kozubowski

distribution embeds the Gaussian and Laplace distributions and has *PDF*

$$\frac{\eta}{2\sigma_g \Gamma(1/\eta)} exp\left[-\left(\frac{|g-\bar{g}|}{\sigma_g}\right)^{\eta}\right],$$

where $\bar{g}$ is the average log growth rate, σ_g is proportional to the standard deviation, and η is a parameter; $\eta = 2$ corresponds to the normal distribution, $\eta = 1$ the Laplace or double exponential.[20]

Data on *g* for all U.S. establishments has been analyzed by Perline *et al*. (2006), shown as a histogram in figure 18 for 1998-99, decomposed into seven logarithmic size classes. Note the vertical axis is *ln*(frequency). In comparison to later years, e.g., 1999-2000, 2000-2001, this distribution is very nearly stationary.

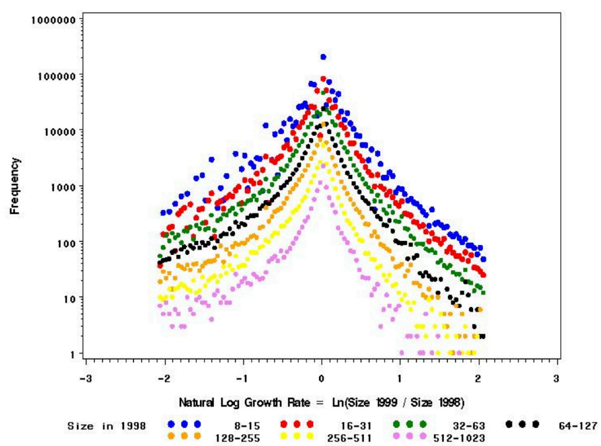

Figure 18*: Histogram of annual g for all U.S. establishments, by size class; source: Census*

Perline *et al*. (2006) find that $\eta \sim 0.60$ for the size 32-63 size class, lesser for smaller firms, larger for bigger ones. The gross statistical features of *g* are:

A. Growth rates *depend* on firm size—small and large firms have different *g*. This means that *Gibrat's law is false*: all firms do *not* have the same *G*.

B. The mode of *g* ~ 0, so mode(*G*) ~ 1, i.e., many firms do not grow.

C. There is more variance for firm decline (*g* < 0) than for growth (*g* > 0), i.e., there is more variability in job destruction than job creation (Davis, Haltiwanger and Schuh 1996), requiring an asymmetric Subbotin distribution

and Podgorski 2001). Schwarzkopf (2010, 2011) argues that *g* is Levy-stable.

[20] For *g* Laplace-distributed, *G* follows the log-Laplace distribution, a kind of double-sided Pareto distribution (Reed 2001), a combination of the power function distribution on (0, 1) and the Pareto on (1, ∞).

(Perline, Axtell and Teitelbaum 2006).

D. Growth rate variance declines with firm size (Hymer and Pashigian 1962, Mansfield 1962, Evans 1987, Hall 1987, Stanley*, et al.* 1996).

There are at least five other well-known regularities concerning firm growth rates that are *not* illustrated by the previous figure:

E. Mean growth is approximately 0;

F. Mean grow rate declines with firm size, and is positive for small firms, negative for large firms (Mansfield 1962, Birch 1981, Evans 1987, Hall 1987, Davis, Haltiwanger and Schuh 1996, Neumark, Wall and Zhang 2011);

G. Mean growth *declines* with age (Evans 1987, Haltiwanger, Jarmin and Miranda 2008);

H. Mean growth *rises* with size, controlling for age (Haltiwanger, Jarmin and Miranda 2011);

I. Growth rate variance *declines* with firm age (Evans 1987).

With these empirical features of firm growth rates as background, figure 19 shows distributions of *g* produced by the model for seven classes of firm sizes, from small (blue) to large (purple) ones.

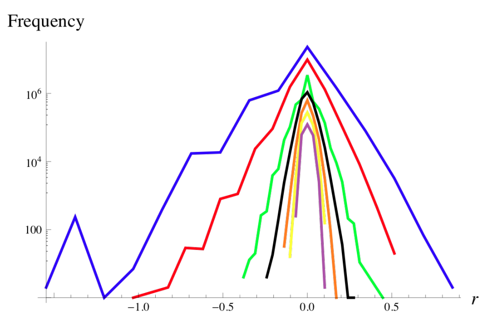

Figure 19*: Distribution of annual g by firm size: 8-15 (blue), 16-31 (red), 32-63 (green), 64-127 (black), 128-255 (orange), 256-511 (yellow), and 512-1023 (purple)*

In this plot we can see at least half of the empirical properties of firm growth: g clearly depends on firm size (*A*), with mode(*g*) = 0 (*B*) and $\overline{g}$ ~ 0.0 (*E*). It is harder to see that there is more variance in firm decline than growth (*C*) but it is the case numerically. Clearly, variance declines with firm size (*D*). Figure 20 shows mean growth rates as a function of firm (a) size and (b) age.

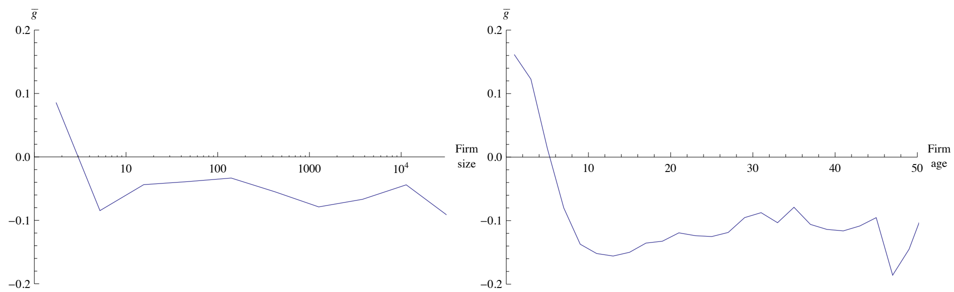

***Figure 20**: Dependence $\bar{g}$ on (a) firm size and (b) firm age, model output*

It is clear from these figures that $\bar{g}$ declines with size (*F*) and similarly for age (*G*). For more than 30 years, since the work of Birch (1981, 1987), economists have debated the meaning of figures like 20a. Specifically, given that small firms are often young and young firms small (e.g., figure 17), it is not clear whether size or age plays the larger role in determining positive growth rates. Haltiwanger and co-workers (2008, 2009, 2011) control for age and argue that it is not small firms that create jobs but rather young ones. The problem with such ‘controls’ for non-montonic relationships is that they mix effects across distinct (size, age) classes. The only actual way to understand the distinct effects of size and age is to show how they each effect $\bar{g}$. This is done in figure 21, where each firm is placed into a (size, age) bin and the average *g* computed locally.

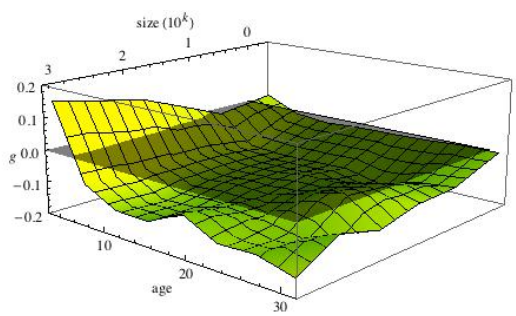

***Figure 21**: Dependence of $\bar{g}$ on firm size and age*

To see precisely whether size or age matters most, a no growth ($\bar{g}$ = 0) plane is superimposed on the model’s $\bar{g}$(size, age). From this we can see that young firms grow the most with a small contribution from small firms.

Firm growth rate variability falls with size (*D*) and age (*I*). Figures 22*a* and *b* show this unconditionally for the model.

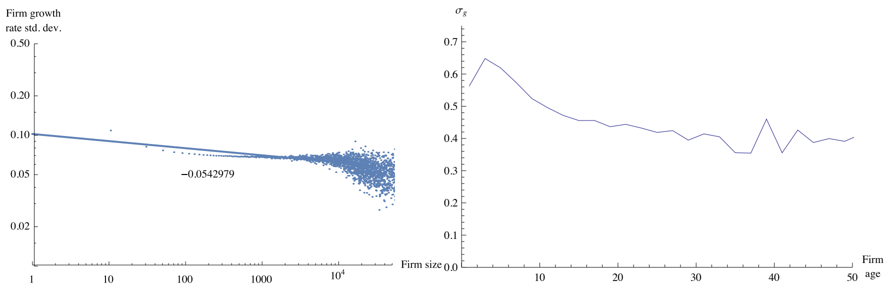

Figure 22*: Dependence of the standard deviation of g on (a) firm size and (b) firm age*

Specifically, the standard deviation of *g* falls with size in figure 21a. Based on central limit arguments one expects this to be proportional to $S^{-\kappa}$, $\kappa = ½$ meaning the fluctuations are independent while $\kappa < ½$ implies they are correlated. Stanley *et al*. (1996) find $\kappa \sim 0.16 \pm 0.03$ for publicly-traded firms (Compustat data) while Perline *et al*. (2006) estimate $\kappa \sim 0.06$ for all U.S. establishments. From the model output $\kappa = 0.054 \pm 0.010$. A variety of explanations for $0 \leq \kappa \leq ½$ have been proposed (Buldyrev*, et al.* 1997, Amaral*, et al.* 1998, Sutton 2002, Wyart and Bouchaud 2002, Fu*, et al.* 2005, Riccaboni, Pammolli, Buldyrev, Ponta and Stanley 2008), all involving firms having internal structure. Note that no such structure exists here, where firms are simply collections of agents, yet dependence of the standard deviation of *g* on size is present nonetheless.

Over any epoch of time some firms grow and others are decline. Expanding firms may shed workers while shrinking ones hire. Figure 23 shows that growing firms hire while suffering separations while declining firms hire even when separations are the norm, much like in the empirical data (Davis, Faberman and Haltiwanger 2006). The 'hiring' line is quite comparable to the empirical result, but the 'separations' line is different—too few separations in the model.

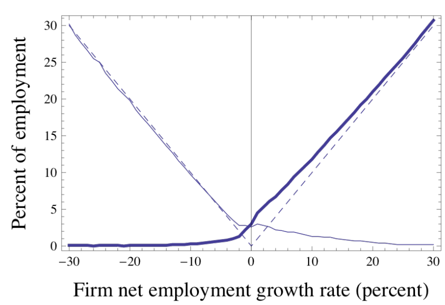

Figure 23*: Labor transitions as a function of firm growth rate, model output*

Having explored firms cross-sectionally, we next turn to the population of agents.

3.4 Agents in Cross-Section: Wages, Job Tenure, Employment

Steady-state worker behavior is quantified here. While each agent's situation adjusts uniquely, at the population level there emerge robust statistical features.

Wage Distribution

While income and wealth are famously heavy-tailed (Pareto 1971 [1927], Wolff 1994), *wages* are less so. A recent empirical examination of U.S. adjusted gross incomes argues that an exponential distribution fits the data below about \$125K, while a power law better fits the upper tail (Yakovenko and Rosser 2009). Figure 24 gives the income distribution from the model.

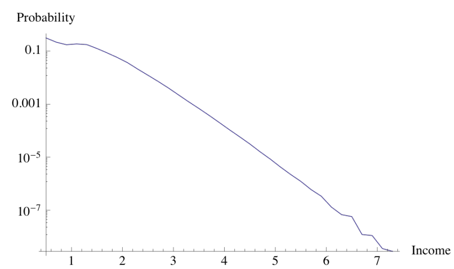

Figure 24*: Wage distribution (arbitrary units)*

Since incomes are nearly linear in this semi-log coordinate system, they are approximately exponentially-distributed.

Job Tenure Distribution

Job tenure in the U.S. has a median of just over 4 years and a mean of about 8.5 years (BLS Job Tenure 2010). The complementary-cumulative distribution for 2010 is figure 25 (points) with the straight line being the model output. As with income, these data are well-approximated by an exponential distribution.

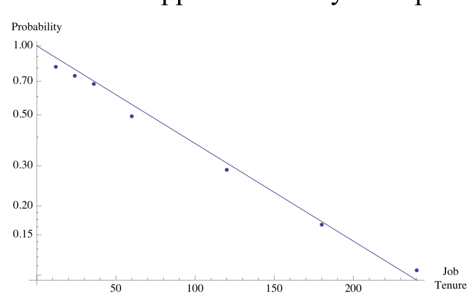

Figure 25*: Job tenure (months) is exponentially-distributed in the U.S. (dots, binned) and in the model (line); source: BLS and author calculations*

The base case of the model is calibrated to make these distributions coincide. That is, the number of agent activations per period is specified in order to make the line go through the points, thus defining the meaning of one unit of time in the model, here a month. The many other dimensions of the model having to do with time—e.g., firm growth rates, ages—derive from this basic calibration.

Employment as a Function of Firm Size and Age

Because the model's firm size distribution by employees is almost exactly right (figure 10a), employment as a function of firm size is also correct. The so-called *Florence median* is the firm size at which half of Americans work in larger and half in smaller firms. It is about 500 for the U.S. and in the model. In figure 26 the dependence of employment on firm age is shown. About half of American workers are in firms younger than 25 years of age. The U.S. data are shown as a counter-cumulative distribution while the model output is shown as points.

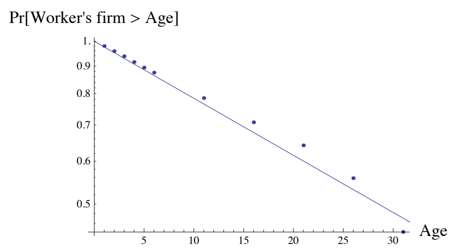

Figure 26*: Counter cumulative distribution of employment by firm age in years in the U.S. (line) and in the model (dots); source: BLS (BDM), available online*

Again there is good agreement between the model and the data. Employment changes by age (Haltiwanger, Jarmin and Miranda 2008) can also be reproduced.

3.5 Inter-firm Worker Migration: The Labor Flow Network

In the model, as in the real world, workers regularly move between jobs. Here the *structure* of such migrations is studied, using a graph theoretic representation of inter-firm labor flows. Let each firm be a node (vertex) in such a graph, and an edge exists between two firms if a worker has migrated between the firms. Elsewhere this has been called the *labor flow network* (Guerrero and Axtell 2013). In figure 27 four properties of this network for the base case of the model are shown. The upper left panel gives the degree distribution, while the upper right is the distribution of edge weights. The lower left panel plots the clustering coefficient as a function of degree, while the lower right panel is the assortativity (average neighbor degree) as a function of degree. These closely reproduce data from Finland and Mexico (Guerrero and Axtell 2013), shown as insets.

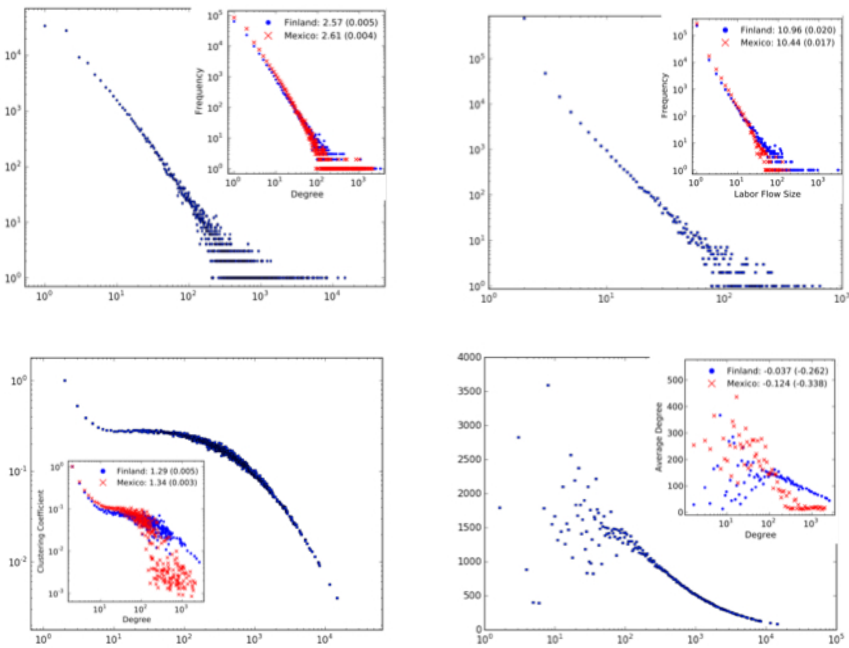

Fig 27: *Properties of the labor flow network (a) degree distribution, (b) edge weight distribution, (c) clustering as a function of degree, and (d) average neighbor degree (assortativity) vs degree*

3.6 Agent Welfare in Endogenous, Multi-Agent Firms

Each time an agent is activated it seeks higher utility, which is bounded from below by the singleton utility. Therefore, it must be the case that all agents prefer the non-equilibrium state to one in which each is working alone—the state of all firms being size one is Pareto-dominated by the dynamical configurations above.

To analyze welfare of agents, consider homogeneous groups of maximum stable size, having utility levels shown in figure 4b, replotted in figure 28.

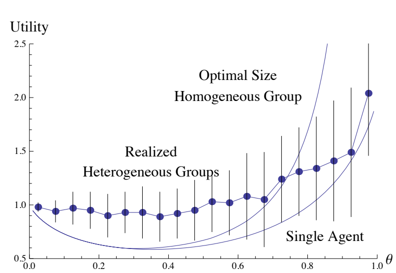

Figure 28: *Utility in single agent firms, optimal homogeneous firms, and realized firms, by θ*

Overlaid on these smooth curves is the cross-section of utilities in realized groups. The main result here is that most agents prefer the non-equilibrium world to the equilibrium outcome with homogeneous groups.

4 Robustness of the Results

In this section the base model of table 1 is varied and the effects described. The main lesson of this section is that, while certain behavioral and other features can be *added* to this model and the empirical character of the results preserved, relaxation of any of the basic specifications of the model, individually, is sufficient to break its connection to the data.

Against this simple model it is possible to mount the following critique. Since certain stochastic growth processes are known to yield power law distributions, perhaps the model described above is simply a complicated way to generate randomness. That is, although the agents are behaving purposively, this may be just noise at the macro level. If agent behavior were simply random, would this too yield realistic firms? We have investigated this in two ways. First, imagine that agents randomly select whether to stay in their current firm, leave for another firm, or start-up a new firm, while still picking an optimal effort where they end up. It turns out that this specification yields only small firms, under size 10. Second, if agents select the best firm to work in but then choose an effort level at random, again nothing like skew size distributions arise. These results suggest that any systematic departure from (locally) purposive behavior is unrealistic.

One specification found to have no effect on the model in the long run is the initial condition. Starting the agents in groups seems to modify only the duration of the initial transient. Next how does the number of agents mater? While the base case of the model has been realized for 120 million agents, figure 29 gives the dependence of the largest firm realized as the population size is varied. The maximum firm size rises sub-linearly with the size of the population.

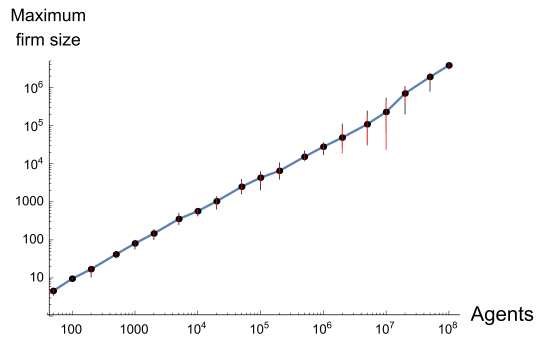

Figure 29: *Largest firm size realized as a function of the number of agents*

Next, consider alternative agent activation schemes. While it is well-known that *synchronous* activation can produce anomalous output (Huberman and Glance 1993), *asynchronous* activation can also lead to subtle effects based on whether agents are activated randomly or uniformly (Axtell, Axelrod, Epstein and Cohen 1996). Moving from uniform to random activation produces slight changes in output.

How does the specification of production matter? Of the three parameters that specify the production function, a, b, and β, as increasing returns are made stronger, larger firms are realized and average firm size increases. For $\beta > 2$, very large firms arise; these are 'too big' empirically.[21]

Are the results presented above robust to different kinds of agent heterogeneity? With preferences distributed uniformly on (0,1) in the base case a certain number of extreme agents exist: those with $\theta \approx 0$ are leisure lovers and those with $\theta \approx 1$ love income. Other distributions (e.g., beta, triangular) were investigated and found to change the results quantitatively but not qualitatively. Removing agents with extreme preferences from the population may result in too few large firms forming, but this can be repaired by increasing β. If agent preferences are *too homogeneous* the model output is qualitatively different from the empirical data. Finally, CES preferences do not alter the general character of the results. Overall, the results are robust for heterogeneous preferences.

Social networks play an important role in the model. In the base case each

[21] If β is sufficiently large the model can occasionally 'run away' to a single large firm.

agent has 2 to 6 friends. This number is a measure of the size of an agent's search or information space, since the agent queries these other agents when active to assess the feasibility of joining their firms. The main qualitative impact of increasing the number of friends is to slow model execution.

However, when agents query *firms* for jobs something different happens. Asking an agent about a job may lead to working at a big firm. But asking a firm at random usually leads to small firms and empirically-irrelevant model output because most firms are small.

How does compensation matter for the results? Pay proportional to effort:[22]

$$U_i^p(e_i;\theta_i,E_{\sim i}) = \left(\frac{e_i}{E}O(E)\right)^{\theta_i}(\omega_i - e_i)^{1-\theta_i}$$

leads to a breakdown in the basic model results, with one giant firm forming. The reason for this is that there are great advantages from the increasing returns to being in a large firm and if everyone is compensated in proportion to their effort level no one can do better away from the one large firm. Thus, while there is a certain 'perfection' in the microeconomics of this pay scheme, it completely destroys all connections of the model to empirical data.

Next consider a mixture of compensation rules, with workers paid partially in proportion to how hard they work and partially based on total output. Calling the *U* of equation (1) U_i^e, a convex combination of utility functions is

$$U_i(e_i) = fU_i^e(e_i) + (1-f)U_i^p(e_i) = \left(\frac{f}{n^{\theta_i}} + \frac{(1-f)e_i^{\theta_i}}{(E_{\sim i}+e_i)^{\theta_i}}\right)\left[O(e_i,E_{\sim i})\right]^{\theta_i}(\omega_i - e_i)^{1-\theta_i}.$$

Parameter *f* moves compensation between 'equal' and 'proportional'. This can be maximized analytically for β = 2, but produces a messy result. Experiments varying *f* show the *qualitative* character of the model is insensitive. Additional sensitivity tests and model extensions are described in the appendix, including variants in which one agent in each firm acts as a residual claimant and hires and fires workers, relaxing the free entry and exit character of the base model.

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[22] Encinosa *et al*. (1997) studied compensation systems empirically for team production environments in medical practices. They find that "group norms" are important in determining pay practices. Garen (1998) empirically links pay systems to monitoring costs. More recent work is Shaw and Lazear (2008).

5 Summary and Conclusions

A model in which individual agents form firms has been analyzed mathematically, realized computationally, and tested empirically. Stable equilibrium configurations of firms *do not exist* in this model. Rather, agents constantly adapt to their economic circumstances, changing jobs when it is in their self-interest to do so. This multi-level model, consisting of a large number of simple agents in an environment of increasing returns, is sufficient to generate macro-statistics on firm sizes, ages, growth rates, job tenure, wages, networks, etc., that closely resemble data, summarized in table 3.

	Datum or data compared	Source	In text
1	Size of the U.S. workforce: 120 million	Census	Table 1
2	Number of firms with employees: ~6 million	Census	Figure 6
3	Number of new firms monthly: ~100 thousand	Fairlie (2012)	Figure 6
4	Number of exiting firms monthly: ~100 thousand	Necessary for steady-state	Figure 6
5	Variance higher for exiting firms than new firms	various	Figure 6
6	Average firm size: 20 employees/firm	Census	Figure 7
7	Maximum firm size: ~1 million employees	Forbes 500	Figure 7
8	Number of job-to-job changes monthly: ~3+ million	Fallick and Fleischman (2004)	Figure 9
9	Number of jobs created monthly: ~2 million	Fallick/Fed spreadsheet	Figure 9
10	Number of jobs destroyed monthly: ~2 million	Fallick/Fed spreadsheet	Figure 9
11	Variance higher for jobs destroyed than jobs created	Davis, *et al.* (1996)	Figure 9
12	Firm size distribution (employees): ~Zipf	Census/Axtell (2001)	Figure 10a
13	Firm size distribution (output): ~Zipf	Census/Axtell (2001)	Figure 10b
14	Aggregate returns to scale: constant	Basu and Fernald (1997)	Figure 11
15	Productivity distribution: Pareto tail	Souma *et al.* (2009)	Figure 12
16	Firm age distribution: Weibull; mean ~14 yrs	Bureau of Labor Statistics	Figure 13
17	Firm survival probability: increasing with age	Bureau of Labor Statistics	Figure 15
18	Joint dist. of firms, size and age: linear in age, log size	Haltiwanger *et al.* (2011)	Figure 16
19	Firm growth rates depend on firm size	various, see text	Figures 17, 18
20	Log firm growth rates (g) are Subbotin-distributed	Stanley *et al.* [1996]	Figure 17, 18
21	Mode(g) = 0.0, many firms do not grow	various, see text	Figure 17, 18
22	More variance for firm decline than firm growth	Davis, *et al.* (1996)	Figure 17, 18
23	Mean of $\bar{g}$ near 0.0, + for small firms, - for large	Birch [1981], others	Fig 19a, 20
24	Variance of g declines with firm size	Stanley *et al.* [1996]	Figure 21a, 23
25	Mean of g declines with firm age	Haltiwanger *et al.* (2011)	Figure 19b, 20
26	Variance of g declines with firm age	Evans (1987)	Figure 21b
27	Mean of g as function of size, age: young firms grow	Haltiwanger *et al.* (2011)	Figure 20
28	Simultaneous hiring and separation	Davis, *et al.* (2006)	Figure 22
29	Wage distribution: exponential	Yakovenko and Rosser (2009)	Figure 23
30	Job tenure dist.: exponential with mean 90 months	Bureau of Labor Statistics	Figure 24
31	Employment vs age: exp. with mean 25 years	Bureau of Labor Statistics	Figure 25
32	Florence (firm size weighted) median: 500 employees	Census	Around fig 25
33	Degree distribution of the labor flow network (LFN)	Guerrero and Axtell (2013)	Figure 28a
34	Edge weight distribution of the LFN	Guerrero and Axtell (2013)	Figure 28b
35	Clustering coefficient vs firm size in the LFN	Guerrero and Axtell (2013)	Figure 28c
36	Assortativity (degree of neighbors) vs firm size, LFN	Guerrero and Axtell (2013)	Figure 28d

Table 3*: Empirical data to which the model output is compared; similar data similarly colored*

Overall, firms are vehicles through which agents realize greater utility than they would by working alone. The general character of these results is robust to many model variations. However, it is possible to sever connections to empirical data with agents who are too homogeneous, too random, or too rational.

5.1 Emergence of Firms, Out of Microeconomic Equilibrium

The main result of this research is to connect an explicit microeconomic model of team formation to emerging micro-data on the population of U.S. business firms. Agent behavior is specified at the micro-level with firms emerging at a meso-level, and the population of firms studied at the aggregate level (figure 30). This micro-meso-macro picture has been created with agent computing, realized at full-scale with the U.S. private sector.[23]

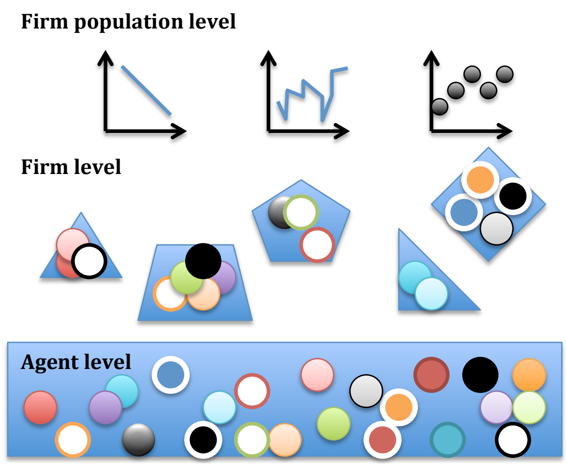

***Figure 30**: Multi-level schematic of firm formation from agents*

However, despite the vast scale of the model, its specification is actually very *minimal*, so spare as to seem rather unrealistic[24]—no product markets are modeled, no prices computed, no consumption represented, no industries appear, and agent behavior is relatively simple. Furthermore, there is no technological change and thus no economic growth—all the dynamics are produced simply through rearrangements of firm personnel to achieve local improvements in the *social* technology of production (Beinhocker 2006). How is it that such a stripped-down model could ever resemble empirical data?

This model works because its *dynamics* capture elements of the real world more closely than conventional models involving static equilibria, either with or

[23] It is folk wisdom that agent models are 'macroscopes,' illuminating macro patterns from the micro rules.
[24] In this it is reminiscent of Gode and Sunder and zero-intelligence traders (Gode and Sunder 1993).

without external shocks. This is so despite the agents being unequipped to figure out optimal multi-period strategies. In defense of such *simple* agents, the environments in which they find themselves are *too complex* for them to compute rational behaviors—each agent's team contains difficult to forecast contingencies concerning co-worker effort levels, the tenure of current colleagues, the arrival of new personnel, and fluctuating outside opportunities.[25] A major finding of this research is that we are able to *neglect* strategic behavior at the firm level—price and quantity-setting, for instance—yet explain many empirical properties of firms. Strategic decisions certainly matter for the fortunes of individual firms, but seem to not be needed to explain the gross properties of the *population* of firms.

More generally, the belief that social (aggregate) equilibria require agent-level equilibria is problematical (Foley 1994, Axtell forthcoming), a classical fallacy of division. The goal of social science is to explain *social* regularities realized at some level 'above' the agent behavioral level. While agent-level equilibria are commonly treated as *necessary*, such equilibria are, in fact, only *sufficient*—macroscopic regularities that have the character of statistical steady-states (e.g., stationary distributions) may result when there do not exist stable agent-level equilibria, as we have seen above. The assumption of homogeneity across levels, whether explicitly made or implicitly followed as a social norm, can be fallacious. Important regularities and patterns may arise at the macro-level without the agent level being in Nash or Walrasian equilibrium. Furthermore, when stable equilibria exist but require huge amounts of time to be realized, one may be better off looking for regularities in long-lived transients. This is particularly relevant to coalition formation games in large populations, where the number of coalitions is given by the unimaginably vast Bell numbers, meaning that anything like optimal coalitions could never be realized during agent lifetimes. Perpetual flux in the composition of groups leads naturally to the conclusion that microeconomic equilibria have little explanatory power.

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[25] Anderlini and Felli (1994) assert the impossibility of complete contracts due to the complexity of nature. Anderlini (1998) describes the kinds of forecasting errors that are intrinsic in such environments.

5.2 From Theories of the Firm to a Theory of Firms

Unfortunately, most extant theories of the firm are steeped in this kind of micro-to-macro homogeneity. They begin innocuously enough, with firms conceived of as being composed of a few actors. They then go on to derive firm performance in response to rivals, strategic uncertainty, information processing constraints, and so on. But these derivations interpret the overall performance of multi-agent groups and organizations in terms of a few agents in equilibrium,[26] and have little connection to the empirical regularities documented above.[27]

There are two senses in which the model described above is a theory of firms. First, from a purely descriptive point of view, the model reproduces the gross features of the U.S. firms, while extant theories of the firm cannot.[28] Nor are most theories sufficiently explicit to be operationalized, mathematically or computationally, their focus on equilibrium leaving behavior away from equilibrium unspecified.[29] In the language of Simon (1976), these theories are substantively rational, not procedurally so. Or, if micro-mechanisms are given, the model is only notionally related to data (e.g., Hopenhayn 1992, Kremer 1993, Rajan and Zingales 2001), or else the model generates the wrong patterns (e.g., Cooley and Quadrini (2001) get *exponential* firm sizes, Klette and Kortum (2004) get *logarithmic* sizes and incorrect dependence of firm growth rate variance on size).

The second sense in which my model is a theory of firms is that agent models are *explanations* of the phenomena they reproduce.[30] In the philosophy of science an explanation is defined with respect to a theory,[31] which has to be general enough to provide explanations of whole classes of phenomena, while not being

[26] Least guilty of this charge is the evolutionary paradigm.

[27] For example, the industrial organization textbooks of both Shy (1995) and Cabral (2000) fail to make any mention whatsoever of firm size, age, or growth rate distributions, nor do they note either the number of firms or the average firm size, either in the U.S. or in other countries!

[28] A variety of models aim for one of these targets, often the firm size distribution (e.g., Lucas 1978, Kwasnicki 1998) and only a handful attempt to get more (Luttmer 2007, 2011, Arkolakis 2013).

[29] I began this work with the expectation of drawing heavily on extant theory. While I did not expect to be able to turn Coase's elegant prose into software line-for-line, I did expect to find significant guidance on the micro-mechanisms of firm formation. These hopes were soon dashed.

[30] According to Simon (Ijiri and Simon 1977: 118): "To 'explain' an empirical regularity is to discover a set of simple mechanisms that would produce the former in any system governed by the latter."

[31] This is the so-called deductive-nomological (D-N) view of explanation; see Hempel (1966).

so vague that it can rationalize all phenomena. Each parameterization of an agent-based model is an instance of a more general agent 'theory'. Executing an instance yields patterns that can be compared to data, thus making it falsifiable.[32]

My 'explanation' for firms is simple: purposive agents in increasing returns environments form quasi-stable coalitions. The ability of agents to move between such transient teams 'arbitrages' away super linear returns. In effect, firms compete for high effort individuals. Successful firms in this environment are ones that can attract and keep productive workers. This model, suitably parameterized, can be compared directly to the emerging micro-data on firms. Today we do not have a mathematical derivation of the aggregate (firm population) properties from the micro (agent behavior) specifications, so for now we must content ourselves with the *discovery* that such firms result from such purposive agents.

This model is a first step toward a more realistic, dynamical theory of the firm, one with explicit micro-foundations. Clearly this approach produces empirically-rich results. We have produced these results computationally. Today computation is used by economists in many ways, to numerically *solve* equations (e.g., Judd 1998), to *execute* mathematical programs, to *run* regressions (e.g., Sala-i-Martin 1997), to *simulate* stochastic processes (e.g., Bratley, Fox and Schrage 1987), or to *perform* micro-simulations (e.g., Bergmann 1990)—all complementary to conventional theorizing. Agent computing enriches these approaches. Like microsimulation, it facilitates heterogeneity, so representative agents (Kirman 1992) are not needed. Unlike microsimulation, it features direct (local) interactions, so networks (Kirman 1997, Vega-Redondo 2007) are natural. Agents possess limited information and are of necessity boundedly rational, since full rationality is computationally intractable (Papadimitriou and Yannakakis 1994). This encourages experimentally-grounded behavioral specifications. Aggregation happens, as in the real world, by summing over agents and firms. Macro-relationships *emerge* and are not limited *a priori* to what the 'armchair economist' (Simon 1986) can first imagine and then solve for analytically. There

[32] In models that are intrinsically stochastic, multiple realizations must be made to find robust regularities.

is no need to postulate the attainment of equilibrium since one merely interrogates a model's output for patterns, which may or may not include stable equilibria. Indeed, agent computing is a natural technique for studying economic processes that are far from (agent-level) equilibrium.

5.3 Economics of Computation and Computational Economics

We have entered the age of *computational synthesis*. Across the sciences, driven by massive reduction in the cost of computing, researchers have begun to reproduce fundamental structures and phenomena in their fields using large-scale computation. In chemistry, complex molecules have their structure and properties investigated digitally before they are manufactured in the lab (Lewars 2011). In biology, whole cell simulation, involving thousands of genes and millions of molecules, has recently been demonstrated (Karr*, et al.* 2012). In fluid mechanics, turbulence has resisted analytical solution despite the governing equations being known since the 19th Century. Today turbulent flows are studied computationally using methods that permit transient internal structures (e.g., eddies, vortices) to arise spontaneously (Hoffman and Johnson 2007). In climate science whole Earth models couple atmospheric and ocean circulation dynamics to study global warming at ever-finer spatiotemporal resolution (Lau and Polshay 2013). In planetary science the way the moon formed after a large Earth impact event has been simulated in great detail (Canup 2012, Cuk and Stewart 2012). In neuroscience high frequency modeling of billions of neurons is now possible, leading to the drive for whole brain models (Markram 2006, 2012).

Surely economics cannot be far behind. Across the social sciences people are utilizing 'big data' in a variety of ways (Lazer*, et al.* 2009). The time has come for a computational research program focused on creating economies in software at full scale with real economies. More than a generation ago an empirically-rich computational model of a specific firm was created and described by Cyert and March (1963) in *A Behavioral Theory of the Firm*. I hope the present work can do for the *population* of U.S. firms what Cyert and March accomplished for an *individual* organization. At this point we have merely scratched the surface of the rich intersection of large-scale agent computing and economics.

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Appendices

A Generalized Preference Specifications

The functional forms of §2 can be relaxed without altering the main conclusions. Consider each agent having preferences for income, I, and leisure, Λ, with more of each being preferred to less. Agent i's income is monotone non-decreasing in its effort level e_i as well as that of the other agents in the group, $E_{\sim i}$. Its leisure is a non-decreasing function of ω_i - e_i. The agent's utility is thus $U_i(e_i; E_i) = U_i(I(e_i; E_{\sim i}), \Lambda(\omega_i - e_i))$, with $\partial U_i/\partial I > 0$, $\partial U_i/\partial \Lambda > 0$, and $\partial I(e_i; E_{\sim i})/\partial e_i > 0$, $\partial \Lambda(e_i)/\partial e_i < 0$. Furthermore, assuming $U_i(I = 0, \cdot) = U_i(\cdot, \Lambda = 0) = 0$, U is single-peaked. Each agent selects the effort that maximizes its utility. The first-order condition is straightforward. From the inverse function theorem there exists a solution to this equation of the form $e_i^* = \max\,[0, \zeta(E_{\sim i})]$. From the implicit function theorem both ζ and e_i^* are continuous, non-increasing functions of $E_{\sim i}$.

Team effort equilibrium corresponds to each agent contributing its e_i^*, and that the other agents are doing so as well, i.e., substituting $E_{\sim i}^*$ for $E_{\sim i}$. Since each e_i^* is a continuous function of $E_{\sim i}$ so is the vector of optimal efforts, $e^* \in [0, \omega]^N$, a compact, convex set. By the Leray-Schauder-Tychonoff theorem an effort fixed point exists. Such a solution constitutes a Nash equilibrium, which is Pareto-dominated by effort vectors having larger amounts of effort for all agents.

For any effort adjustment function $e_i(t+1) = h_i(E_{\sim i}(t))$, such that

$$\frac{dh_i(E_{\sim i})}{dE_{\sim i}} = \frac{\partial h_i(E_{\sim i})}{\partial e_j} \leq 0,$$

for all $j \neq i$, there exists an upper bound on firm size. Under these circumstances the Jacobian matrix retains the structure described in §2.3, where each row contains N-1 identical entries and a 0 on the diagonal. The bounds on the dominant eigenvalue derived in §2.3 guarantee that there exists an upper bound on the stable group size, as long as the previous inequality is strict, thus establishing the onset of instability above some critical size.

B Generalized Compensation and Nash Stability

It was asserted in section 4 that proportional or piecemeal compensation breaks our basic results. What it does is dramatically reduce the incentive problems of team production. To see this we redo figure 1 for this compensation function, as shown in figure A.1

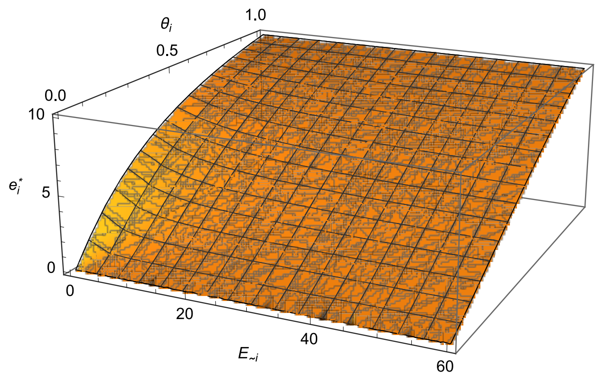

Figure A.1*: Dependence of e_i^* on $E_{\sim i}$ and θ_i for a = 1, b = 1, ω_i = 10*

Note that there is no longer a region of zero effort. We next compute the Jacobian matrix and evaluate its elements as the size of the group increases. This is shown in figure A.2.

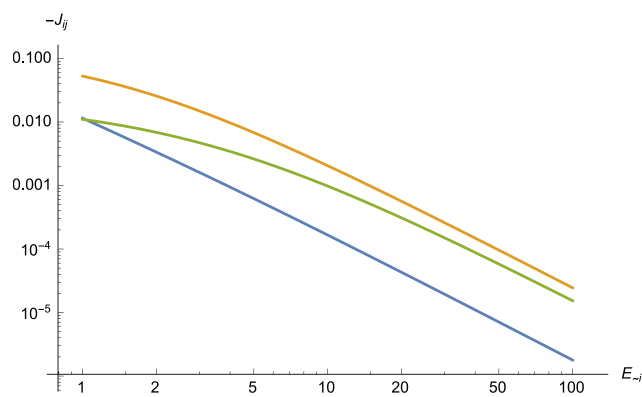

Figure A.2*: Dependence of the elements of the Jacobian matrix on $E_{\sim i}$ for a = 1, b = 1, and ω_i = 1, for three values of θ_i (0.1, 0.5, and 0.9)*

The values decline sufficiently rapidly (note the *log-log* coordinates) that no instability will be induced by the dynamical effort level adjustments of the agents to one another, no matter how large the group.

For mixtures of compensation we recover the general properties of equal compensation. The way that effort, e_i^*, depends on $E_{\sim i}$ and θ_i for $f = ½$ is shown in figure A.2. Note the region of zero effort for agents with low preference for income.

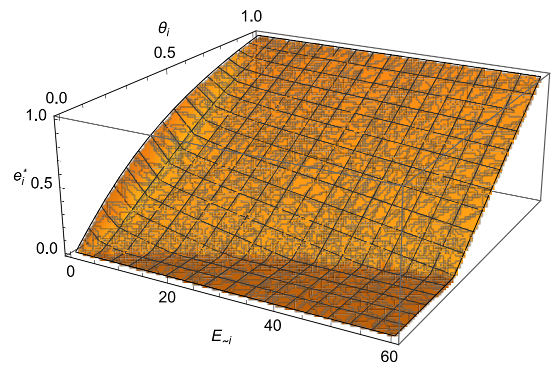

***Figure A.3**: Dependence of e_i^* on $E_{\sim i}$ and θ_i for a = 1, b = 1, ω_i = 10 and f = 1/2*

For this mixture of compensation policies the eigenvalues of the Jacobian matrix can be computed numerically for various values of

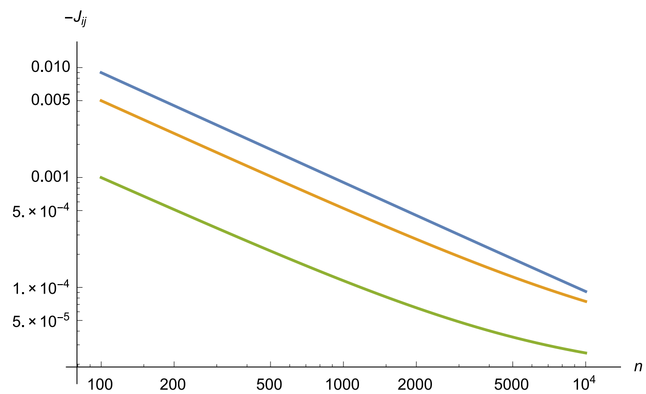

***Figure A.4**: Dependence of the elements of the Jacobian matrix on n for a = 1, b = 1, ω_i = 1, f = ½, and $E_{\sim i}$ = 100, for three values of θ_i (0.1 (blue), 0.5 (orange), and 0.9 (green))*

While these values still decline as an approximate power law, they do so sufficiently slowly that it becomes possible to produce eigenvalues outside the unit circle, particularly for large *n*, since the matrix entries begin plateauing then.

C Sensitivity to 'Sticky' Effort Adjustment

In the base model agents adjust their effort levels to anywhere within the feasible range [0, ω]. A different behavioral model involves agents making only small changes from their current effort level each time they are activated. Think of this as a kind of prevailing *work ethic* within the group or *individual habit* that constrains the agents to keep doing what they have been, with small changes.

Experiments have been conducted for each agent searching over a range of 0.10 around its current effort level: an agent working with effort e_i picks its new effort from the range $[e_L, e_H]$, where $e_L = max(0, e_i - 0.05)$ and $e_H = min(e_i + 0.05, 1)$. This slows down the dynamics somewhat, yielding larger firms. This is because as large firms tend toward non-cooperation, this kind of sticky effort adjustment dampens the downhill spiral to free riding. I have also experimented with agents who 'grope' for welfare gains by randomly perturbing current effort levels, yielding similar results.

D Extension: Stabilizing Effect of Agent Loyalty

In the basic model an agent moves immediately to a new firm when its subjective evaluation is that it will be better off by doing so. Behaviorally, this seems implausible for certain kinds of workers, especially those who feel some *loyalty* to their firm. The formulation of agent loyalty used here involves agents not changing jobs right away, as soon as they figure out that they can do better elsewhere. Rather, they let χ better job opportunities arrive before separating from their current firm. Think of an agent's χ as a kind of counter. It starts off with some value and each time the agent determines there are higher payoffs elsewhere but does not leave its firm the value of χ declines by 1. When $\chi = 0$ the next preferable position it can find it takes and χ is reset. The base case of the model corresponds to no loyalty, that is, $\chi = 0$.

I have experimented with homogeneous and heterogeneous Λs, in the range from [0, 10]. Even a modest amount of loyalty reduces worker turnover and firm volatility, especially in large firms, and increases job tenure, firm age, and firm lifetime, holding other parameters constant. Increasing loyalty makes large firms bigger while reducing labor flows. In order to maintain the close connection of the model output to empirical data in the presence of agent loyalty it would be necessary to recalibrate the model, something reserved for future work.

E Extension: Hiring

One aspect of the base model is very unrealistic: that agents can join whatever firms they want, as if there is no barrier to getting hired by any firm. The model can be made more realistic by instituting local hiring policies.

Let us say that one agent in each firm does all hiring, perhaps the agent who founded the firm or the one with the most seniority. We will call this agent the 'boss'. A simple hiring policy has the boss compare current productivity to what would be generated by the addition of a new worker, *assuming that no agents adjust their effort levels*. The boss computes the minimum effort, $\phi E/n$, for a new hire to raise productivity as a function of a, b, β, E and n, where ϕ is a fraction:

$$\frac{aE + bE^{\beta}}{n} < \frac{a\left(E + \phi\frac{E}{n}\right) + b\left(E + \phi\frac{E}{n}\right)^{\beta}}{n+1} = \frac{aE\left(1+\frac{\phi}{n}\right) + bE^{\beta}\left(1+\frac{\phi}{n}\right)^{\beta}}{n+1}. \quad \text{(A.8)}$$

For $\beta = 2$ this can be solved explicitly for the minimum ϕ necessary

$$\phi_{*} = \frac{-n(a+2bE) + \sqrt{n^2(a+2bE)^2 + 4bEn(a+bE)}}{2bE}.$$

For all values of ϕ_{*} exceeding this level the prospective worker is hired. For the case of $a = 0$, (A.8) can be solved for any value of β: $\phi_{*} = n\left(\frac{n+1}{n}\right)^{1/\beta} - n$; this is independent of b and E. The dependence of ϕ_{*} on β and n is show in Table A.3.

$n \backslash \beta$	**1.0**	**1.5**	**2.0**	**2.5**
1	1.0	0.59	0.41	0.32
2	1.0	0.62	0.45	0.35
5	1.0	0.65	0.48	0.38
10	1.0	0.66	0.49	0.39
100	1.0	0.67	0.50	0.40

Table A.3: Dependence of the minimum fraction of average effort on firm size, *n,* and increasing returns parameter, β

As n increases for a given β, ϕ_{*} increases. In the limit of large n, ϕ_{*} equals $1/\beta$. So with sufficient increasing returns the boss will hire just about any agent who wants a job! These results can be generalized to hiring multiple workers.

Adding this functionality to the computational model changes the behavior of individual firms and the life trajectories of individual agents but does not substantially alter the overall macrostatistics of the artificial economy.

F Extension: Effort Monitoring and Worker Termination

In the base model, shirking goes completely undetected and unpunished. Effort level monitoring is important in real firms, and a large literature has grown up studying it; see Olson (1965), the models of mutual monitoring of Varian (1990), Bowles and Gintis (1998), and Dong and Dow (1993), the effect of free exit (Dong and Dow 1993), and endowment effects (Legros and Newman 1996); Ostrom (1990) describes mutual monitoring in institutions of self-governance.

It is possible to *perfectly* monitor workers and fire the shirkers, but this breaks the model by pushing it toward static equilibrium. All real firms suffer from imperfect monitoring. Indeed, many real-world compensation systems can be interpreted as ways to manage incentive problems by substituting reward for supervision, from efficiency wages to profit-sharing (Bowles and Gintis 1996). Indeed, if incentive problems in team production were perfectly handled by monitoring there would be no need for corporate law (Blair and Stout 1999).

To introduce involuntary separations, say the residual claimant knows the effort of each agent and can thus determine if the firm would be better off if the least hard working one were let go. Analogous to hiring we have:

$$\frac{aE + bE^{\beta}}{n} < \frac{a\left(E - \phi\frac{E}{n}\right) + b\left(E - \phi\frac{E}{n}\right)^{\beta}}{n-1} = \frac{aE\left(1 - \frac{\phi}{n}\right) + bE^{\beta}\left(1 - \frac{\phi}{n}\right)^{\beta}}{n-1}$$

Introducing this logic into the code there results unemployment: agents are terminated and do not immediately find another firm to join. Experiments with terminations and unemployment have been undertaken and many new issues are raised, so we leave full investigation of this for future work.

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