EQUILIBRIUM TRADE IN AUTOMOBILE MARKETS*

KENNETH GILLINGHAM †
YALE UNIVERSITY

FEDOR ISKHAKOV ‡
UNIVERSITY OF NEW SOUTH WALES

ANDERS MUNK-NIELSEN §
UNIVERSITY OF COPENHAGEN

JOHN RUST ¶
GEORGETOWN UNIVERSITY

BERTEL SCHJERNING ‖
UNIVERSITY OF COPENHAGEN

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Abstract

We present a computationally tractable dynamic equilibrium model of the automobile market where new and used cars of multiple types (e.g. makes and models) are traded by heterogeneous consumers. The prices and quantities of used cars are determined endogenously to equate supply and demand for all car types and traded vintages, along with the ages at which cars are scrapped as part of the equilibrium solution. The model allows for transactions costs, taxes, and flexible specifications of car characteristics, consumer preferences, and heterogeneity. We demonstrate the existence of multiple equilibria in these markets and focus on Pareto dominant "maximal equilibria" where the ages at which cars are scrapped are maximized. We use a "small open economy" version of our model where prices of new cars are fixed on the world market to analyze the effect of changes in car tax policy on government revenues, auto prices, driving, and consumer welfare in Denmark. We find significant welfare gains from a budget neutral replacement of the ultra high (180%!) tax on new cars in favor of a 30% gas tax. We also use the model to analyze olipopolistic price setting in the new car market, where car producers take into account consumer substitution possibilities between new and used versions of different makes and models of cars as well as between different types of new cars. We use our model to analyze the welfare and effects of a "merger to monopoly" in the new car market. In both the tax reform and merger cases, we show the high price of new cars has a significant negative welfare impact that falls especially hard on poorer consumers, and drives substantial number of them out of the car market.

KEYWORDS: secondary markets, trade, consumer heterogeneity, transactions costs, dynamic discrete choice, stationary equilibrium, Markov chains, invariant distributions

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†Yale University, Forestry & Environmental Studies, Department of Economics, School of Management, 195 Prospect Street, New Haven, CT 06511, kenneth.gillingham@yale.edu

‡University of New South Wales, Centre of Excellence in Population Ageing Research, Kensington NSW 2052, Australia, f.iskhakov@unsw.edu.au

§University of Copenhagen, Department of Economics and Centre for Computational Economics (CCE), Øster Farimagsgade 5, building 26, DK-1353, Copenhagen K, Denmark, anders.munk-nielsen@econ.ku.dk

¶**Corresponding Author:** Georgetown University, Department of Economics, Georgetown University, Washington, DC, jr1393@georgetown.edu

‖University of Copenhagen, Department of Economics and Centre for Computational Economics (CCE), Øster Farimagsgade 5, building 26, DK-1353 Copenhagen K, Denmark, bertel.schjerning@econ.ku.dk

1 Introduction

We present a computationally tractable dynamic equilibrium model of the automobile market where new and used cars of multiple types (e.g. makes and models) are traded by heterogeneous consumers. The prices and quantities of used cars are determined endogenously to equate supply and demand for all car types and traded vintages, and the ages at which cars are scrapped is also determined endogenously as part of the equilibrium. The model allows for transactions costs, taxes, and flexible specifications of car characteristics, consumer preferences and heterogeneity. We demonstrate the existence of multiple equilibria in these markets and characterize Pareto dominant "maximal equilibria" where the ages at which cars are scrapped are maximized.

Our approach is based on a dynamic discrete choice model of ownership and trading of automobile where a key part of our specification of consumer heterogeneity includes additive idiosyncratic extreme value components to the utilities from the different possible choices individuals face in the market. Preference shocks have a clear economic motivation, since they can be interpreted as representing idiosyncratic random components to the cost of maintaining an existing car, as well as idiosyncratic variation in the costs of search and in the prices and transactions costs of different cars the consumer might want to buy. These shocks constitute a fundamental source of gains from trade that rationalize the existence of secondary markets. Though our model assumes symmetrically informed buyers and sellers, it allows for non-stochastic components of transactions costs that can include taxes and costs of vehicle inspections and warranties that are standard ways markets deal with and insure against "lemons problems" that might otherwise kill off secondary markets for the reasons noted in the seminal work of Akerlof (1970). By varying the scale of these additive extreme value preference shocks, we show how reductions in consumer heterogeneity reduce gains from trade that utlimately kills off secondary markets when there are sufficiently large transactions costs due to taxes, information asymmetries, or other trade frictions.

We believe this is a promising framework for empirical applications, since the extreme value specification results in logit (or nested logit) functional forms for the conditional choice probabilities for the decisions to keep or trade diffent types and ages of vehicles. The logit formulas are attractive because they have the flexibility to match the rich types of trading patterns we observe in the data, including consumers who choose not to own cars (i.e. who choose the "outside good") and "brand loyalty" and "brand switching" behaviors. In addition, we show that the choice probabilities are smooth functions of car prices and that excess demands are appropriately aggregated linear combinations of these choice probabilities and so are also smooth functions of prices. This smoothness implies that we can use a robust and efficient algorithm, Newton's method, to solve consumers' dynamic trading strategies as well as equilibrium prices. We introduce an iterative algorithm that finds converges to a maximal equilibrium in a finite number of iterations that also serves as an economically motivated equilibrium selection mechanism that endoenously determines the ages at which different types of cars are scrapped in equilibrium. The flexibility and tractability of our model makes it promising for use in empirical applications including structural estimation and policy forecasting.

We formulate our model and define equilibrium in an infinite horizon stationary environment where we can use the machinery of Markov processes to describe trading behavior and characterize the vehicle holdings of different types of consumers as invariant distributions to certain Markov chains. These Markov chains reflect both the trading of vehicles, as well as the aging and the impact of stochastic accidents that result in premature scrappage of some vehicles. The stationary equilibrium concept results in a very compact and elegant description of equilibrium. However our definition of equilibrium can be extended to non-stationary environments where macroeconomic shocks lead to fluctuations in auto trading, scrapping, and new car sales that result in "waves" in the age distribution of automobiles. We can also extend our equilibrium analysis to an "overlapping generations" framework where individual lifepan is finite. Since the notation and analysis to define and calculate equilibrium in these various extensions is substantially more complicated, we have opted to introduce our approach in its simplest form in a stationary, infinite horizon setting where we can define and characterize the equilibrium very elegantly and simply.

This work was driven by the need for large scale, practical numerical equilibrium models to evaluate car tax policy reforms in Denmark, which has one of the highest taxes on new and used cars in the world. The Danish registration tax is a piece-wise linear function of the price of a new passenger vehicles, equal to a percentage τ_1 of the vehicle price for cars that sell for less than a threshold P_1 and a percentage τ_2 for vehicles that sell for more than P_1. Prior to 2016 the values of (P_1, τ_1, τ_2) were $(81700, 1.05, 1.8)$, and electric vehicles were exempted from the registration tax. Since 81700 DKK is approximately only \$12000, the majority of new cars sold in Denmark are subject to this extremely high 180% new car tax. The registration tax also applies to used car sales. Discontent over the magnitude of the registration tax lead to reforms, and in 2016 the tax schedule was changed to $(185100, 0.85, 1.5)$, though electric and hybrid cars became subject to a registration tax but at a rate equal to only 20% of the registration tax for a gas powered passenger car with an additional 10000 DKK deduction through 2019.[1] The Danish government also imposes fuel taxes and ownership taxes, but the registration tax is by far the most significant component of car taxes.

The Danish registration tax can be viewed as a substantial government-imposed transactions cost that inhibits trade in automobiles. However reducing it is not easy, since it accounts for between 30 and 50 billion DKK in tax revenue annually, i.e. about 4 to 7 percentage of Danish tax revenues. Danes are also concerned about the environmental impact of automobiles including CO_2 emissions and traffic congestion, and alternative tax policy changes under discussion include raising the fuel tax, or imposing a usage tax on VKT, vehicle kilometers travelled. In this paper we use a stylized, calibrated version of our model that is fit to roughly match a few key moments of car prices and holdings in Denmark to evaluate the effects of alternative tax changes currently under discussion on tax revenues, vehicle prices and market share, driving, and consumer welfare. In separate work, Gillingham, Iskhakov, Munk-Nielsen, Rust and Schjerning (2019), we structurally estimate a more realistic and detailed overlapping generations model of vehicle holding and trading behavior using microdata Danish Register data to obtain much more accurate and disaggregated predictions of the welfare effects of tax policy changes for different types of households in Denmark.

In section 2 we summarize the existing literature on numerical dynamic equilibrium models of the automobile market to clarify how our contribution relates to this larger body of work. In section 3 we present a dynamic equilibrium model of trading in new and used markets for multiple types of automobiles by multiple types of consumers with transactions costs. In section 4 we use the model to analyze changes in Danish tax policy relating to cars using a simple, stylized and calibrated model of the Danish car market. In section 5 we extend the model of section 3 to allow endogenous determination of the prices of new cars, under the assumption that competing new car manufacturers set Bertrand-Nash equilibrium prices that consider not only substitution possibilities between various types of new cars, but also the natural substitution between new and used cars and the option of keeping a currently held vehicle or trading an existing vehicle for another used vehicle instead of buying a new one. In section 6 we discuss extensions and other applications of the framework introduced in this paper.

[1] The tax on electric vehicles is scheduled to gradually rise to parity with gas/diesel powered vehicles by 2022. Hybrid cars are taxed at a rate intermediate between the rate for electric and gas/diesel vehicles, though this advantage will also phase out by 2022.

2 Previous Models of Trade in Automobile Markets

The influential paper by Berry, Levinsohn and Pakes (1995) is the dominant approach to modeling equilibrium in the automobile market. They consider the demand for new cars in isolation, ignoring the presence of the secondary market for used cars using a static discrete choice model for new cars with a quasi-linear specification for utility. This is the basis of the well known BLP approach for estimating preferences for new cars using aggregate market share data when there are instrumental variables to deal with potential econometric price endogeneity due to unobserved car attributes that affect new car prices in equilibrium. However since used cars are natural substitutes for new cars and keeping the current used car or trading it for another used car is an alternative to buying a new car, it is not clear that a framework that ignores these obvious substitution possibilities can provide reliable estimates of the demand for new cars. In particular, we show that in a dynamic model the prices of *all traded cars* enter consumers' value functions *nonlinearly* even when car prices enters a consumer's current period utility linearly. Since the BLP estimator depends on the assumption of linearity, it cannot deal with price endogeneity issues in structural estimation of dynamic models of auto demand. For this reason, BLP may not be a reliable approach for estimating consumer preferences and new car production costs, and their incomplete static model of new car demand may not be an adequate framework to describe how new car prices and sales are determined.

Rust (1985d) and Esteban and Shum (2007) were the first to tackle the more challenging problem of solving for a full equilibrium in both the primary and secondary markets for automobiles. Rust studied the simultaneous determination of price and durability by a monopolist new car producer, and Esteban and Shum studied oligopolistic pricing of competing new car producers. Both studies explicitly account for the substitution possibilities afforded by a secondary market. Esteban and Shum explain why "The durability of cars and the existence of a secondary market have important competitive implications for new-car producers. The secondary market introduces, in the form of used cars, a large number of (imperfect) substitutes to the new cars produced each period, which limits the market power of each producer. In turn, rational firms recognize that their current production will reach the secondary market in the future and, by lowering prices in those markets, will erode future profits" (p. 332). To make progress both of these studies assume stationarity and zero transactions costs. This converts a dynamic problem into a simpler static choice problem: consumers trade each period for their most preferred vehicle, which could be either a new or used car of any type.[2]

[2]Esteban and Shum (2007) also assume quality ladder preferences that further simplifies the choice problem.

Our key contribution is to introduce a dynamic equilibrium model of automobile trading that incorporates transactions costs. This results in more realistic patterns of trade in automobiles between heterogeneous consumers where the typical behavior will be to "buy and hold" rather than to trade cars every period. The implication of this is that only an endogenously determined subset of the full population of consumers will be involved in buying and selling new and used cars each period. Our dynamic equilibrium model enables us to characterize the subset of consumers who are most likely to trade, as well as different subsets of the population who are more likely to buy and hold different types of cars. Since the pattern of trade and vehicle holdings is endogenous and determined in equilibrium, a dynamic equilibrium perspective is important for studying questions of identification and how to control for endogeneity and unobserved heterogeneity when it comes to the econometric estimation of these models.

We build on a substantial theoretical and empirical literature that focused on modeling equilibrium in secondary markets for automobiles, taking the price of new cars as given. The earliest work that we are aware of on numerical equilibrium modeling of automobile markets is a series of papers by Manski (1980), Manski and Sherman (1980), and Manski (1983) that inspired subsequent research on micro-econometrically estimable equilibrium models of the automobile market. Their work was the first to introduce theoretical models of equilibrium in secondary markets for cars that could be numerically solved for prices and quantities and used for policy forecasting of a wide range of policies of interest. Another key contribution was Berkovec (1985) who microeconometrically estimated and numerically solved a large scale equilibrium model of the new and used car markets, where he posited exogenously specified supply functions for new cars as a function of new car prices. He modeled household choices of different car types and ages using a nested logit model that allows for patterns of correlation in the unobserved components of the utility of different cars to capture patterns of similarity in the observed characteristics of 13 different car classes (e.g. luxury cars, compact cars, vans, pickups, etc) and estimated the model for households that own 1, 2, or 3 cars using the 1978 U.S. National Transportation Survey.

Berkovec constructed an "expected demand function" for vehicles of different ages and classes by summing the estimated discrete choice probabilities for cars of each age and class. He defined an equilibrium to be a vector of prices (with one price for each possible age and price of car) that equates the expected demand for vehicles of each car age and type to the supply of these vehicles, net of scrappage.[3] Berkovec used Newton's method to compute the

[3]Berkovec used a probabilistic model of vehicle scrappage due to Manski and Goldin (1983) where the probability a vehicle is scrapped is a decreasing function of the difference between the second-hand price of the car (net of any repair costs) and an

equilibrium prices in the market, by finding a zero to a system 131 nonlinear equations for different vehicle class/age price categories. At the time Berkovec did his work, inverting the 131×131 Jacobian matrix of excess demands in order to implement Newton's method was a computational challenge, unlike today.[4] Though Berkovec did not present the equilibrium prices calculated from his model, he concluded that "Overall, the simulation model forecasts appear to do reasonably well for the 1978-1982 period." (Berkovec (1985), p. 213).

The contributions of Manski and Sherman and Berkovec were extremely advanced given the limits of computing power at the time, and still represent the closest point of departure and template for our own work in this area. However their work was based on short run, static equilibrium holding models of the car market. That is, they assumed that consumers make a repeated series of *static* discrete choices over the optimal type and age of car to hold, assuming quasi-linear utility functions where the utility of each car age and car type is evaluated net of the cost of maintaining the car and the expected depreciation on the car, where the latter equals the discounted difference between the price of purchasing the desired car at the start of the period, less the expected discounted resale value of the car at the end of the period.

Implicit in the static discrete choice formulation is the assumption that consumers only keep their vehicle for a single period, so that at the end of each period consumers trade their current vehicle for their most preferred vehicle type and age. Rust (1985c) formulated and solved a fully dynamic (i.e. infinite horizon) model of trading in the automobile market under the assumption of zero transactions costs and complete (i.e. symmetric) information. He assumed the state of a car is captured by its odometer value x_t which evolves according to an exogenous Markov process representing variable usage of cars with transition probability $\Phi(x_{t+1}|x_t)$ that represents stochastic usage and deterioration of vehicles. Since x_t fully captures the state of a car and is observable by both parties in a transaction, Rust's analysis avoided "lemons problem" information asymmetries of the type analyzed in the seminal work of Akerlof (1970) that can potentially kill off the secondary markets for cars.

Rust assumed that consumers have quasi-linear preferences for autos, i.e. nonlinear preferences for cars of different physical conditions, as captured by the vehicle odometer x, but utility for all other consumption c is additively separable and linear in c, so preferences take the form $u(x,\tau)+c$ where τ indexes potentially heterogenous preferences over how quickly utility declines as a function of car condition x. He showed that when there are no transactions costs and

exogenously specified scrap value for the vehicle. This implies that, except for random accidents, there is very little chance that new cars are scrapped, but the probability a used car is scrapped increases monotonically with the age of the car.

[4] Berkovec showed that the Jacobian matrix had special structure he called "identity outer product" that enabled him to invert the Jacobian via inverting a smaller 48×48 matrix and doing some additional matrix vector multiplications.

the economy is in a stationary equilibrium (i.e. when there are no macro shocks or other time-varying factors altering the prices or quantities of vehicles in the market), the optimal trading strategy for all consumers is to trade their cars each period for the most preferred age/condition of car $x^*(\tau)$. Rust proved that a unique stationary equilibrium exists in this market where the equilibrium price function $P(x)$ is the unique fixed point to a contraction mapping. Scrappage is endogenously determined in this model as the smallest odometer value γ where the price of a car $P(\gamma)$ equals its scrap value $\underline{P}$, under the assumption of an infinitely elastic demand for cars at an exogenously specified scrap value $\underline{P}$. The equilibrium price function $P(x)$ coincides with "shadow prices" to a planning problem where a social planner chooses an optimal stopping time (the first time when x_t exceeds the optimal scrappage threshold γ) that maximizes the expected discounted utility of the representative consumer for holding an infinite sequence of cars, taking the cost of new cars $\overline{P}$ and their scrap value $\underline{P}$ as given.

Rust showed that when there are no transactions costs the optimal choice of car by consumer of type τ, $x^*(\tau)$, can be reformulated as a static utility maximization problem similar to what Manski and Berkovec assumed, where consumers choose a car to maximize a quasi-linear utility that trades off the value of newness less the cost of owning a car x which equals the sum of expected maintenance costs and a "rental value" $R(x)$ equal to expected depreciation, $R(x) = P(x) - \beta EP(x)$ (where β is the consumer's discount factor and $EP(x)$ is the expected resale price at the end of the period for a car whose odometer is x at the start of the period). Vehicle "quantities" are given by an *equilibrium holdings distribution* $F(x)$ which is an invariant distribution to the Markov transition probability $\Phi(x_{t+1}|x_t)$. In a stationary equilibrium the economy is in "flow equilibrium" so that the fraction of new cars purchased each year equals $1 - F(\gamma)$, the fraction of used cars which are scrapped. Using renewal theory, he showed that $F(x)$, the fraction of cars with odometer less than or equal to x, is a ratio of mean first passage times: i.e. $F(x)$ equals the mean time for a new vehicle's odometer to first exceed x divided by the mean time it first exceeds the scrappage threshold γ.[5]

However zero transactions costs is an unrealistic assumption that implies that consumers trade every period for their preferred condition $x^*(\tau)$. This is something we definitely do not observe in reality. When there are transactions costs (which are distinct from *trading costs*, i.e. the difference between the price of a car x a consumer wishes to buy, $P(x)$, less the price $P(x')$ of the car x' that the consumer wishes sell), Rust (1985c) showed that the optimal strategy

[5]Rust (1985a) showed this theory provides a good approximation to the joint distribution of ages and odometer values in the US economy using data from the 1970s. He also showed that for a range of plausible utility functions for consumers, the stationary equilibrium resulted in *convex price functions* $P(x)$, which implies the rapid early depreciation for new cars and the slower depreciation for older cars that we observe in most auto markets.

generally involves keeping the current car for multiple periods. In a stationary market the optimal trading strategy in the presence of transactions costs will take the form of an "(S,s) strategy" reminiscent of optimal inventory theory. That is, each consumer type τ will have two thresholds $(\underline{x}^*(\tau),\overline{x}^*(\tau))$ where $\underline{x}^*(\tau) < \overline{x}^*(\tau)$ and $\underline{x}^*(\tau)$ is the condition of the optimal replacement car that a consumer will choose whenever he/she trades their for a new one. $\overline{x}^*(\tau)$ is the *replacement threshold* or odometer threshold where it is optimal to trade the current car in condition x for a replacement car in condition $\underline{x}^*(\tau)$. When transactions costs are zero, then $\overline{x}^*(\tau) = \underline{x}^*(\tau) = x^*(\tau)$ and it is optimal to trade for the optimal car $x^*(\tau)$ every period.

Consumers who are sufficiently rich or who have a sufficiently strong preference for "newness" will replace their used car with a brand new one, $\underline{x}^*(\tau) = 0$, so there will be a mass of consumers whose preferred replacement vehicle are new cars. However not all of these consumers will choose to buy new cars each period: only the subset whose odometer values x exceed their replacement thresholds $\overline{x}^*(\tau)$ will trade. In the case of a homogenous consumer economy, positive transactions costs will completely kill off trading in the secondary market for autos: all consumers will prefer to follow a "buy and hold" transaction where all will buy a brand new vehicle $\underline{x}^* = 0$ and sell their cars at the socially optimal scrappage threshold $\overline{x}^*(\tau) = \gamma$ which coincides with the consumer's optimal replacement threshold when consumers are homogeneous and secondary markets do not exist.

Establishing the existence of a stationary equilibrium in a heterogeneous agent economy in the presence of transactions costs is challenging. Consider a consumer of type τ who desires to buy a relatively new but not brand new car $\underline{x}^*(\tau) > 0$. When there are transactions costs there is no guarantee that will be some other consumer τ' willing to sell their car this early, i.e. it might be that for any price function P the mimimum value of $\overline{x}^*(\tau')$ over all consumer types $\tau' \neq \tau$ exceeds $\underline{x}^*(\tau)$. Thus, when transactions costs are sufficiently high there may be "no trade regions" sufficiently close to $x = 0$ regardless of how slowly car prices depreciate there. Accounting for these regions and how they change as P changes enormously complicates the problem of calculating an equilibrium or even proving that one exists.

Using advanced methods from functional analysis (e.g. the Fan-Glicksburg fixed point theorem), Konishi and Sandfort (2002) established the existence of a stationary equilibrium in the presence of transactions costs under certain conditions. Their proof shows that it is possible for the equilibrium price function $P(x)$ to adjust to prevent the coordination failures of the type discussed above, i.e. where some consumer type τ' wishes to buy some sufficiently new car $\underline{x}^*(\tau')$ but no other consumer type τ is willing to sell their used car to that consumer. However to our knowledge, nobody has been able to calculate equilibria with transactions costs in the infinite-dimensional model considered by Konishi and Sandfort (2002), i.e. where the state of a car is given by its odometer value, x, which can take a continuum of possible values.

Stolyarov (2002) advanced the literature by assuming that the state of a car can be summarized by its age a which can take only a finite number of values, $a = 0, 1, 2, \ldots, \overline{a}$, where $\overline{a}$ is the age where cars are scrapped. By treating car states as discrete, Stolyarov followed the work of Manski, Sherman and Berkovec, but he also assumed there is a continuum of consumers similar to the work by Rust and Konishi and Sandfort. Stolyarov showed that his continuous uni-dimensional parameterization of consumer heterogeneity with quasi-linear preferences provides sufficient gains from trade to make it possible to numerically solve the fixed point problem that characterizes the equilibrium price function.[6] When the state of a car is discrete, the fixed point problem becomes a finite-dimensional problem where the fixed point is the vector of equilibrium prices $P \in R^{\overline{a}}$ where $\overline{a}$ is the scrappage age for cars in his economy.

The discrete choice model of Stolyarov (2002) differs from the one proposed by Manski and Berkovec by using a more restrictive uni-dimensional index of consumer heterogeneity τ as the source of gains from trade between consumers in the auto market. All consumers agree that a car of age a provides x_a units of service, where x_a declines with age. A consumer of type τ has a quasi-linear utility function given by $u_\tau(a, c) = \tau x_a + c$ where a is the age of the car the consumer owns and c is the consumption of other goods. Stolyarov also assumed probabilistic proportional transactions costs: with probability $\alpha \in (0, 1)$ a seller of a car of age a receives a fraction $(1 - \lambda)p_a$ of the secondary price p_a, and with probability $(1 - \alpha)$ the seller receives the full price p_a, where $\lambda \in (0, 1)$ is a proportional transactions cost parameter. This specification implies that the optimal trading strategy remains of the $(\underline{a}^*(\tau), \overline{a}^*(\tau))$ form, where $\underline{a}^*(\tau)$ is the preferred car age consumer τ trades for whenever the age a of their current car exceeds $\overline{a}^*(\tau)$ or whenever the realized value of transactions costs is zero.

Stolyarov defined an equilibrium by first deriving the steady state holding distributions $h_\tau(a)$ for each consumer type τ, which has support on the interval $(\underline{a}^*(\tau), \overline{a}^*(\tau))$. Demand for cars of age $\underline{a}^*(\tau)$ occurs whenever a) the consumer draws a zero transaction cost, or b) the consumer's car first exceeds the replacement threshold $\overline{a}^*(\tau)$. In either case they trade their

[6]Stolyarov did not prove that the excess demand function for cars is a continuous function of prices, not did he provide a proof of the existence of equilibrium in his model. It is not hard to see that excess demand cannot be a continuous function of prices if there are only a finite number of possible consumer types τ, so having a continuum of types appears to be a necessary condition for existence of equilibrium. Stolyarov's equation (13) for excess demand involves a univariate integration over consumer types, but this integral will generally not have a closed-form solution and must be approximated via numerical integration. However numerical integration, such as Gaussian quadrature, amounts to solving the problem using a finite number of consumer types, so in all likelihood Stolyarov found a price vector that made excess demand "small" but not exactly equal to zero.

car of age a for a replacement car of age $\underline{a}^*(\tau)$. Supply of cars of age a occurs whenever the consumer draws a zero transactions cost shock, or a exceeds the optimal replacement threshold $\overline{a}^*(\tau)$. The fractions of car supplied and demanded by consumer τ can be calculated using the stationary holding distribution $h_\tau(a)$. Total supply and demand is then calculated by integrating over the distribution of consumer types τ, $f(\tau)$. Prices are numerically calculated to set $ED(P) = 0$ where $P \in R^{\overline{a}}$ and $\overline{a}$ is an exogeneously specified age at which cars become useless, and thus provide zero services $x_{\overline{a}} = 0$.[7]

Using a calibrated version of his model, Stolyarov showed that "that the probability of resale is nonmonotonic in the age of the good. Trade volume is relatively low in the very beginning and in the middle of a goods life. This result helps explain observed variations of resale rates across vintages for the U.S. market of used cars." (p. 1390). Stolyarov's approach has also been extended by Gavazza, Lizzeri and Roketskiy (2014) to allow households to own up to two cars using a two-dimensional specification of consumer heterogeneity. They find that "Calibration of the model successfully matches several aggregate features of the US and French used-car markets. Counterfactual analyses show that transaction costs have a large effect on volume of trade, allocations, and the primary market." (p. 3668).

Our model of equilibrium in the auto market is basically similar to Stolyarov (2002) except that we follow the earlier work of Manski and Berkovec by using a multi-dimensional extreme value specification of consumer heterogeneity as a key component in a hierachical specification of heterogeneity that includes both time-varying idiosyncratic preference shocks (i.e. the extreme value error terms in the model) as well as very flexible specifications for time invariant heterogeneity or types τ, with a population distribution that can can have either continuous or finite support. The special properties of the extreme value distribution result in continuous formulas for choice probabilities even in the case where there is no other time invariant heterogeneity. We will show that the continuity of choice probabilities in the price vector P implies the existence of equilibrium via the Brouwer fixed point theorem. More importantly, we will show that the excess demand function for used cars in our model, $ED(P)$, is a smooth (i.e. continuously differentiable) function of P that enables us to rapidly and accurately calculate equilibrium prices by solving the system of nonlinear equations $ED(P) = 0$ by Newton's method. This makes our approach very attractive for use in empirical work and policy modeling.

There is a close connection between models of auto trading that incorporate transactions

[7] Stolyarov (2002) defined an operator $\Lambda(P) = P + ED(P)$ and actually computed P as an approximate fixed point of Λ.

costs and models that emphasize information asymmetries, such as Akerlof (1970). There are "lemons laws" in many countries that require sellers to compensate buyers for defects or problems in a car that were not disclosed and negotiated on at the time of sale. Further, a majority of used car transactions take place through intermediaries — car dealers — that are under additional legal and reputational constraints not to sell lemons to their customers. When a consumer trades in a used vehicle to a dealer, the dealer will typically inspect a vehicle to find any problems and then repair them prior to selling the car to another customer. Many used car dealers also provide limited term warranties on the cars they sell, resulting in additional protections to buyers. The costs of the mechanical inspections and repairs and warranties that mitigate the informational asymmetries result in transactions costs that are often borne by the seller, especially when the seller is a car dealer. As a result, it is not clear that informational asymmetries are actually the most serious problem inhibiting trade in used cars. Instead, the combination of lemons law disclosure restrictions and warranties show in the form of transactions costs.

Hendel and Lizzeri (1999) were among the first to study equilibria in auto markets with and without asymmetric information. In a simple two period framework, they find that adverse selection problems do not necessarily kill off the secondary market: "We find that the used market never shuts down, the volume of trade can be large, and distortions are lower than previously thought." (p. 1097). Further they note that "The empirical evidence on the presence of adverse selection in used cars is inconclusive." (p. 1112). Further, they find that it is difficult to distinguish empirically between the predictions of models with asymmetric information and those with transactions costs on the observed prices of cars. They extend their model to the case of two types of cars, Fords and Hondas and compare the implications of two hypotheses: 1) the two brands differ in the extent to which they suffer from adverse selection from 2) a symmetric information model with transactions costs where the two brands differ with respect to their relative rates of physical depreciation. The conclude that "The interpretation of these two rough sources of evidence suggested by our model is that the steeper price declines for American cars are better explained by the difference in depreciation rates than by the difference in the severity of adverse selection problem." (p. 1108). In light of this, it seems justifiable to ignore the complexities created in dynamic models of trade with asymmetric information and use transactions costs to capture trade frictions when considering how to move forward in modeling the dynamics of trading in auto markets.

3 Dynamic Equilibrium in the Automobile Market

In this section we introduce a dynamic model of an automobile market with multiple car types with heterogeneous consumers. We adopt an extreme value specification of time-varying consumer heterogeneity that results in a nested logit specification for choice probabilities similar to Berkovec (1985). We then add additional layers of time-invariant and time-varying observed and unobserved heterogeneity among consumers, resulting in a rich, flexible framework that is also tractable for use in empirical applications. Time-varying extreme value preference shocks is the key to the tractability of our approach, since it results in excess demand functions that are easy to compute smooth functions of the prices of cars allowing us to compute equilibria using Newton's method, just as Berkovec did.

The key difference between our model and Berkovec's is that our model is fully dynamic and allows for transactions costs, and this implies that consumers will generally follow "buy and hold" strategies similar to those characterized by Stolyarov (2002). However Stolyarov's specification only allows for uni-dimensional heterogeneity, and this results in restrictive patterns of trading among consumers. In particular, all consumers of the same type behave identically, always buying the same replacement car $\underline{a}^*(\tau)$ whenever they trade. The much richer extreme value distributed time-varying heterogeneity of the consumers in our model results in positive probabilities of keeping their current car or buying any age or type of car, or the outside good. The more flexible holding and trading behavior implied by our model makes it more attractive for use in empirical work where any observed pattern of trades will have positive probability, whereas the uni-dimensional heteorgeneity in Stolyarov's model will predict that many observed trading sequences will have zero probability of occuring in any equilibrium. As we discuss below, the *IID* extreme value preference shocks in our model can be interpeted as capturing a host of idiosyncratic factors affecting consumer decisions such as unexpectedly high or low maintenance costs or search costs, and deviations in transactions prices from the expected equilibrium or "blue book" values. Thus, equilibrium prices in our model can be interpreted as *expected transaction prices,* and the additive extreme value shocks can be viewed as including deviations or residuals in actual transaction prices from the expected prices calculated in our model that arise due to idiosyncratic features of used cars that the parties to the transaction observe, but which we as the econometrician do not observe.

We start by defining an equilibrium in the secondary market, taking the prices of new cars as given and setting the price of each age of traded used cars to clear the market. Our model can therefore be viewed as an equilibrium for a "small open economy" where prices of new cars are determined in the world market. We believe this is an appropriate way to model the automobile market in Denmark, for example. Given the extremely high Danish registration tax that we noted in the introduction, prices of new cars can be viewed as policy instruments. In section 4 we analyze the welfare and budgetary impacts of reducing the registration tax in favor of higher fuel and road usage taxes. In section 5 we show how new car prices can be fully endogenized in a dynamic oligopoly model where competing car manufacturers set new car prices to maximize profits, taking into account the responses of their competitors and the reactions of consumers whose auto holding, replacement, and scrappage decisions also change when an auto company changes the price of its new vehicles.

It is helpful to start by considering a simple special case of the general model: equilibrium in a market with a single car type and homogeneous consumers and no outside good. In this case the problem becomes particularly simple and we show equilibrium prices are the solution to a system of linear equations. We show that the results of Rust (1985c) apply, so equilibrium prices and the age at which cars are scrapped can be interpreted as "shadow prices" to a particular social planning problem, and where the scrappage age $\overline{a}$ for cars is the optimal stopping boundary to an optimal stopping problem. Though this equilibrium may seem unrealistic for the reasons noted in the previous section, we will show in section 3.2 that the prices from the homogeneous agent equilibrium actually provide very good starting points for the iterative solution (via Newton's method) of the much more complicated system of *nonlinear* equations that characterize equilibrium in the dynamic multiple car type heterogeneous agent economy.

3.1 Homogeneous consumer economy, single car type, no transactions costs

Consider a stationary, infinite-horizon economy where initially there is only one type of car (though of different ages) and consumers live forever and maximize expected discounted utility with common discount factor $\beta \in (0,1)$. Cars are a necessity (i.e. the disutility of not owning a car is sufficiently large that all consumers will choose to own a car) but we assume the marginal utility of an additional car is sufficiently small that no consumer would want to own more than a single car. Following the literature, we will assume consumers have a common quasi-linear utility function for cars so that the utility of a consumer who owns a car of age a and has income y that can be used for consumption of other goods is $u(a) + \mu y$, where $\mu > 0$ is the "marginal utility of money". The parameter μ is a simple way to capture "income/wealth effects" in this model. As we will see below, consumers who have high values of μ can be interpeted as "poor consumers" since the cost of buying a new car will involve a high utility opportunity cost in terms of forgone consumption of other goods. Thus, μ will constitute a crucial parameter governing a consumer's willingness to pay for a car. We assume that the expected cost of maintaining a car and any utility from driving is subsumed into the function $u(a)$. In our analysis of Danish car tax policies in section 4 we will consider models where fuel prices affect desired driving and $u(a)$ can be viewed as an indirect utility function that reflects the optimal level of driving given the age of the car and the time-invariant fuel price (which is supressed for notational simplicity).

When there are no transactions costs, we show below that it is optimal for the consumer to trade every period for their most preferred vehicle age $a^* \in \{0,1,\dots,\overline{a}-1\}$ where $\overline{a}$ is the age at which cars in this economy are scrapped. For technical reasons we will elaborate on more below we will assume that consumers are not allowed to buy and drive cars that are destined for the scrap yard (i.e. we assume that consumers are not allowed to own any car of age $\overline{a}$ or older). In fact in Denmark and many other countries the government does have annual or semi-annual safety inspections of cars and will not allow cars that are in sufficiently poor condition to be driven. Thus, we should imagine $\overline{a}$ to be a sufficiently old age for a car where it is no longer economic to undertake the maintenance and repair expenses to insure the car can pass an annual government safety inspection. Since we also assume that old cars do not provide utility to car collectors, cars that cannot be driven are effectively useless to consumers and will be scrapped.

Suppose consumers can buy new cars at an exogenously fixed price $\overline{P}$ and there is an infinitely elastic demand for cars for scrap metal at price $\underline{P} < \overline{P}$. In addition, assume there is a secondary market where used cars are traded and consumers can buy or sell a car of age a at a price $P(a)$ with no transtions cost, where $a \in \{1,\dots,\overline{a}-1\}$. Note that consumers do incur *trading costs* of $P(a^*) - P(a)$ when they sell their car of age a to buy a desired car of age a^*. However it is important to note that we assume there are no taxes or additional transactions costs in addition to this trading cost.

The Bellman equation for a consumer who owns a car of age $a \in \{1,\dots,\overline{a}-1\}$ is given by

$$V(a) = \max\left[u(a) + \beta V(a+1), \max_{d\in\{0,1,\dots,\overline{a}-1\}} [u(d) - \mu[P(d) - P(a)] + \beta V(d+1)]\right]. \quad (1)$$

Notice that we consider the decision at the *start of each period* and assume that a consumer always owns a car, so at this point we exclude the possibility of selling an existing car and not replacing it with another one. We also exclude the possibility of owning a new car $a = 0$ at the start of the period. The reason this is excluded is that if the consumer had purchased a new car in the previous period, that car would be of age $a = 1$ at the start of the current period. While we do allow the consumer to buy a new car (by selling their existing car and purchasing a new car at price $P(0) = \overline{P}$), due to our timing convention and definition of a as the age of the *current car, at the start of the current period, before the consumer has made a decision on whether to trade it or keep it,* it is not possible for the age of an existing car to be $a = 0$. However we assume trading occurs instantaneously, so just after consumers have made their trades it will be possible to see a positive fraction of consumers who purchase brand new cars. Thus at the start of each period, prior to trading, the possible values of the age state variable are $a \in \{1, 2, \ldots, \overline{a}\}$.

Given our assumption that consumers are prohibited from driving cars that are age $\overline{a}$ or older, the Bellman equation for a consumer with a car of age $a \geq \overline{a}$ is given by

$$V(a) = \max_{d \in \{0, \ldots, \overline{a}-1\}} [u(d) - \mu[P(d) - P(a)] + \beta V(d+1)] \,. \tag{2}$$

It is easy to see from the Bellman equation (1) that is always optimal for consumers to trade for a preferred car age a^*, equal to the value of d that attains the maximum in the sub-maximization problem in the second term in brackets in the Bellman equation (1). This implies that

$$V(a) = u(a^*) - \mu[P(a^*) - P(a)] + \beta V(a^* + 1) \tag{3}$$

and in particular,

$$V(a^*) = u(a^*) + \beta V(a^* + 1) \tag{4}$$

and

$$V(a^* + 1) = V(a^*) - \mu[P(a^*) - P(a^* + 1)]. \tag{5}$$

so these equations imply that

$$V(a^*) = \frac{u(a^*) - \beta\mu[P(a^*) - P(a^* + 1)]}{1 - \beta}. \tag{6}$$

Note that the expression for $V(a^*)$ in equation (6) is for a consumer who already owns the optimal age car, a^*. If the consumer trades each period for a car of age a^* the present value of utility from owning a future sequence of cars of age a^* is $u(a^*)/(1 - \beta)$. If the first period is period $t = 0$ then the consumer does not incur any cost of buying car a^* in period $t = 0$ since the consumer already owns it. But starting in period $t = 1$ and continuing for every $t \in \{1, 2, 3, \ldots\}$ the consumer will incur future trading costs in order to trade back to their preferred car age a^*. These trading costs are $[P(a^*) - P(a^* + 1)]$ in every period $t \in \{1, 2, 3, \ldots\}$ and the present value of these costs (translated from dollars to utils) equals $\beta\mu[P(a^*) - P(a^* + 1)]/(1 - \beta)$. The extra factor β is necessary to discount these trading costs back to period $t = 0$. So this is the intuitive explanation for the expression of the value function in equation (6).

Now consider a consumer with a car of age $a \neq a^*$. This consumer will have to sell this car in period $t = 0$ and buy the optimal age car a^*. The trading cost for this is $[P(a^*) - P(a)]$ (which could be negative if $a < a^*$). The Bellman equation (1) implies that

$$V(a) = V(a^*) - \mu[P(a^*) - P(a)] = \frac{u(a^*) - \mu[P(a^*) - \beta P(a^* + 1)]}{1 - \beta} + \mu P(a). \tag{7}$$

This equation tells us that the discounted utility of a consumer from buying their optimal choice of car a^* equals the discounted stream of utilities (net of maintenance cost) $u(a^*)/(1-\beta)$, less the discounted stream of *depreciation costs* (converted to utils) $\mu[P(a^*) - \beta P(a^* + 1)]/(1-\beta)$.

If all consumers have the same discount factor β and have homogeneous preferences, then in equilibrium all consumers must be indifferent between holding any of the available ages of vehicles. That is, there should be no consumer who has a strict preference for any particular age $a^* \in \{0, 1, \dots, \overline{a} - 1\}$. Given that we assume new car prices and scrap prices are exogenously fixed at values $\overline{P}$ and $\underline{P}$, respectively, there are only $\overline{a} - 1$ "free prices" left to equilibrate supply and demand for used cars of ages $a \in \{1, \dots, \overline{a} - 1\}$, and their prices are $(P(1), \dots, P(\overline{a} - 1))$. In a homogeneous consumer economy, these prices must adjust to make consumers indifferent about holding any of the ages of vehicles, $d \in \{0, 1, \dots, \overline{a} - 1\}$. From equation (7) we see that the discounted utility from a policy of trading for a car of age d in every period $t \in \{0, 1, 2, \dots\}$ is given by $U(d)$ given by

$$U(d) = \frac{u(d) - \mu[P(d) - \beta P(d + 1)]}{1 - \beta} \tag{8}$$

If consumers are indifferent between all available ages of vehicles, then $U(d) = K$ for all $d \in \{0, 1, \dots, \overline{a} - 1\}$ for some constant K, or

$$u(d) - \mu[P(d) - \beta P(d + 1)] = K(1 - \beta), \tag{9}$$

for $d \in \{0, 1, \dots, \overline{a} - 1\}$. These indifference restrictions imply a system of $\overline{a} - 1$ linear equations in the $\overline{a} - 1$ unknowns $P(1), \dots, P(\overline{a} - 1)$. This system can be written in matrix form as

$$X \times P = Y \tag{10}$$

where $P' = (P(1),\dots,P(\overline{a}-1))$ and X is the $\overline{a}-1 \times \overline{a}-1$ matrix given by

$$X = \begin{bmatrix} -\mu(1+\beta) & \mu\beta & 0 & \cdots & 0 & 0 \\ \mu & -\mu(1+\beta) & \mu\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \mu & -\mu(1+\beta) & \mu\beta \\ 0 & 0 & \cdots & 0 & \mu & -\mu(1+\beta) \end{bmatrix}, \tag{11}$$

and Y is a $\gamma-1 \times 1$ vector given by

$$Y = \begin{bmatrix} u(0)-u(1)-\mu\overline{P} \\ u(1)-u(2) \\ \cdots \\ u(\overline{a}-3)-u(\overline{a}-2) \\ u(\overline{a}-2)-u(\overline{a}-1)-\mu\beta\underline{P} \end{bmatrix}. \tag{12}$$

Notice that we do not impose a) monotonicity or b) the restriction that $P(a) \in [\underline{P},\overline{P}]$ on the solution P to the linear system (10). Thus, we need to check if the solution has these properties. If it does, it is an equilibrium since the price vector results in all consumers being indifferent between holding any one of the available vehicles that are traded in the new or secondary markets, $a \in \{0,1,\dots,\overline{a}-1\}$. The equilibrium "quantities" are the holdings of vehicles of these different ages. It is easy to see that without any accidents or "endogenous scrappage" of cars prior to the scrappage threshold age $\overline{a}$, then the equilibrium or steady state age distribution of cars will be uniform on the interval $\{0,\dots,\overline{a}-1\}$, so that a fraction $1/\overline{a}$ of the total vehicle stock will be of age a at the *beginning of each period* just *after* the consumers have made their trading decisions (the distribution will be uniform on the set $\{1,\dots,\overline{a}\}$ at the start of the period, but just *before* they have traded their vehicles).

This implies in particular that (assuming all consumers hold just one car) that the fraction $1/\overline{a}$ of the population will buy a new car each period, and the corresponding fraction will scrap their cars, so the market will be in "flow equilibrium". It will also be in "stock equilibrium" since the fact that consumers are indifferent about which age vehicle they own and hold, they can be arranged so that their demand for the different ages is also uniform, matching the supply. Thus there will be zero excess demand for any vehicle age $a \in \{0,1,\dots,\overline{a}-1\}$ for the price function given above.

Note that multiple equilibria are possible in this model. That is, we can find different values of $\overline{a}$ and different corresponding price vectors P (one for each conjectured value of $\overline{a}$) that satisfy the linear system (10) and are monotonically decreasing from $\overline{P}$ to $\underline{P}$. These equilibria can be Pareto-ranked, with the equilibria corresponding to larger values of $\overline{a}$ being Pareto-preferred by consumers to equilibria with smaller values of $\overline{a}$. That is, consumer welfare is lower in equilibria where cars are scrapped "prematurely". However $\overline{a}$ cannot be increased to arbitrarily large values for a fixed utility function $u(a)$. Eventually for large enough $\overline{a}$, the solution P to the linear system (10) is no longer monotonically decreasing from $\overline{P}$ to $\underline{P}$ and thus no longer constitutes an equilibrium.

Thus, there is a range of potential homogeneous consumer equilibria that can be ranked by the scrappage age $\overline{a}$, i.e. the smallest age a for which $P(a) = \underline{P}$. We will now show that there is a largest possible value of $\overline{a}$ and we will refer to the corresponding equilibirum as the *maximal equilibrium.* We now show that this maximal value of $\overline{a}$ equals the scrappage threshold a social planner would choose in an appropriately specified social planning problem. However there is a sense in which the other Pareto-subdominant equilibria in this model are artificial — a consequence of the government restriction that consumers cannot drive cars once they reach the scrappage age $\overline{a}$. If this restriction is removed, we will show that the Pareto-dominanted values of $\overline{a}$ that are less than the maximal value cannot be supported in equilibrium because they are dominated by "autarky" — i.e. consumers can obtain higher utility by buying new cars and keeping them until they scrap them at an optimal age of their choosing. This optimal scrappage age will be γ, the scrappage age in the Pareto dominant maximal equilibrium.

The consumer's autarky problem turns out to be mathematically equivalent to the social planning problem. In the latter the social planner determines the optimal age to scrap cars in order to maximize consumer utility, in the absence of a secondary market. Thus the social planning problem is equivalent to a *regenerative stopping problem* — the planner chooses the optimal age threshold at which to replace their current car with a brand new one to maximize the consumer's expected discounted utility over an infinite horizon. The value function $W(a)$ to this problem is given by

$$W(a) = \max\left[u(0) + \beta W(1) - \mu[\overline{P} - \underline{P}], u(a) + \beta W(a+1)\right]. \tag{13}$$

It is easy to see from the Bellman equation above that $W(0) = u(0) + \beta W(1)$ and thus, any consumer who is "endowed" with a brand new car would never immediately replace it with another new one since this would involve the additional replacement cost $\overline{P} - \underline{P}$. However if

$u(a)$ is decreasing sufficiently rapidly there will be a finite age, $\overline{a}$, for which we have

$$W(\overline{a}) = u(0) + \beta W(1) - \mu[\overline{P} - \underline{P}] \geq u(\overline{a}) + \beta W(\overline{a}+1) \tag{14}$$

and we define $\overline{a}$ as the smallest integer satisfying the inequality above and it is easy to show that for this $\overline{a}$ we have

$$W(\overline{a}) = W(0) - \mu[\overline{P} - \underline{P}]. \tag{15}$$

Using the value function to the social planning problem, we can define a *shadow price function* $P(a)$ by

$$P(a) = \overline{P} - [W(0) - W(a)]/\mu. \tag{16}$$

Notice that this shadow price function statisfies $P(0) = \overline{P}$, $P(\overline{a}) = \underline{P}$, and $P(a)$ is monotonically declining in a for the values of a for which $W(a)$ is monotonically decreasing in a, which is the set of $a \in \{0, 1, \ldots, \overline{a}-1\}$. However it is not hard to see from the Bellman equation (13) that for $a < \overline{a}$ we have

$$W(a) = u(a) + \beta W(a+1), \tag{17}$$

which simply says that it is optimal for the consumer to keep their car if its age is younger than the optimal scrappage age $\overline{a}$. However using this condition, it is easy to verify that the shadow prices (16) make consumers indifferent between all car ages $a \in \{0, 1, \ldots, \overline{a}-1\}$. Specifically, we want to show that

$$u(a) - \mu[P(a) - \beta P(a+1)] = K \tag{18}$$

for $a \in \{0, 1, \ldots, \overline{a}-1\}$ where $\overline{a}$ is the optimal scrapping threshold from the solution to social planning problem (13) and $P(a)$ is given by the “shadow prices” in equation (16). Notice that equation (16) implies that

$$P(a) - \beta P(a+1) = [\overline{P} - W(0)/\mu](1-\beta) + [W(a) - \beta W(a+1)]/\mu. \tag{19}$$

However the Bellman equation (13) implies that for all $a < \overline{a}$ we have

$$W(a) - \beta W(a+1) = u(a). \tag{20}$$

Substituting equation (20) into equation (19) we obtain

$$P(a) - \beta P(a+1) = [\overline{P} - W(0)/\mu](1-\beta) + u(a)/\mu \tag{21}$$

and substituting this expression into the left hand side of the indifference condition (18) we obtain

$$u(a)-\mu[P(a)-\beta P(a+1)] = u(a)-\mu[\overline{P}-W(0)/\mu](1-\beta)-u(a) = [W(0)-\mu\overline{P}](1-\beta). \quad (22)$$

Since $W(0)-\mu\overline{P}$ does not depend on a, it follows that the shadow prices in equation (16) do result in a homogeneous consumer equilibrium as claimed.

We can also solve for the value function $W_{\overline{a}}(a)$ for a potentially suboptimal policy of "keep the car until age $\overline{a}$ and then scrap it" for some other arbitrary scrappage age $\overline{a}$ not necessarily equal to the optimal scrappage age from the solution to equation (14) above. We have

$$\begin{aligned} W_{\overline{a}}(a) &= u(a)+\beta W_{\overline{a}}(a+1), \quad a \in \{0,1,\dots,\overline{a}-1\} \\ W_{\overline{a}}(\overline{a}) &= W_{\overline{a}}(0)-\mu[\overline{P}-\underline{P}]. \end{aligned}$$

This is a linear system of equations that has a unique solution $W_{\overline{a}}$ and clearly we have $W_{\overline{a}}(a) \leq W(a)$ since $W(a)$ constitutes the optimal car replacement policy. It is easy to follow the same steps as above to show that we can define shadow prices $P_{\overline{a}}(a) = \overline{P} - [W_{\overline{a}}(0) - W_{\overline{a}}(a)]/\mu$ analogous to the shadow price function (16) defined in terms of the optimal value function W. By construction, these shadow prices will make the consumer indifferent between choosing to buy different ages of cars available in the secondary market and hence are the same as the solutions to equation (10). However for choices of $\overline{a}$ that exceed the socially optimal scrappage threshold, the solutions will not necessarily be monotonically decreasing in a and prices may dip below $\underline{P}$ for some values of a, and for this reason these solutions will not constitute valid equilibria. For values of $\overline{a}$ that are less than the socially optimal scrappage threshold, the value functions $W_{\overline{a}}(a)$ will be monotonically decreasing in a and hence the shadow prices $P_{\overline{a}}(a)$ will also be monotonically decreasing in a (and thus lie in the interval $[\underline{P},\overline{P}]$ since by construction we have $P_{\overline{a}}(0) = \overline{P}$ and $P_{\overline{a}}(\overline{a}) = \underline{P}$), however even these candidate solutions cannot be equilibria in the secondary market absent an arbitrary restriction or government regulation that mandates that all cars of age $\overline{a}$ or older must be scrapped. Without this restriction, then the "autarky solution" is for consumers to keep cars until the optimal scrappage age $\overline{a}$ given by the smallest integer satisfying inequality (14), and no consumer would choose to participate in a secondary market equilibrium that involved scrapping cars earlier than this socially optimal scrappage age $\overline{a}$.

Theorem 0 *The consumer's discounted utility in the Pareto dominant homogeneous consumer equilibrium equals the welfare the consumer obtains from the solution to the social planning* *problem. That is,*

$$W(a) = V(a), \quad a = 1, \ldots, \overline{a} \tag{23}$$

where $\overline{a}$ is the optimal scrappage threshold and W is the value function from the solution to the social planning problem (13), and V is the solution to the consumer problem (1) in the homogeneous consumer equilibrium where prices P are given by the shadow prices in equation (16) and also for $a = 0$ we have

$$u(0) + \beta V(1) = W(0) \tag{24}$$

so that the discounted utility of a consumer in the homogeneous consumer equilibrium who is endowed with a new car at time $t = 0$ is the same as the welfare the consumer would receive under the social planner problem, which is equivalent to a "buy and hold" strategy where the consumer never trades the car until its age exceeds the scrappage threshold $\overline{a}$, at which age the consumer scraps their existing car and buys a new one.

It is also easy to see that the following Corollary holds.

Corollary *Let $\overline{a}$ be an integer that does not equal the optimal scrappage threshold given by the solution to the social planning problem in (13) and (14). Let the shadow price function $P_{\overline{a}}(a)$ be given by*

$$P_{\overline{a}}(a) = \overline{P} - [W_{\overline{a}}(0) - W_{\overline{a}}(a)]/\mu, \quad a \in \{0, 1, \ldots, \overline{a}\} \tag{25}$$

where the value function $W_{\overline{a}}(a)$ is given by the solution to equation (23). Define the value function $V_{\overline{a}}(a)$ as the value a consumer would obtain in a hypothesized secondary market where the trading strategy involves selling their current car of age a for a replacement car of age $d \in \{0, 1, \ldots, \overline{a} - 1\}$ each period, and the consumer is indifferent about which age car d to replace their current car with

$$V_{\overline{a}}(a) = \frac{u(d) - \mu[P_{\overline{a}}(d) - \beta P_{\overline{a}}(d+1)]}{1 - \beta} + \mu P_{\overline{a}}(a). \tag{26}$$

Then we have:

$$V_{\overline{a}}(a) = W_{\overline{a}}(a), \quad a \in \{0, 1, \ldots, \overline{a}\} \tag{27}$$

From the Corollary it is now straightforward to see why the existence of Pareto suboptimal equilibria depend on a legal restriction that forces consumers to scrap cars prior to the optimal scrappage threshold $\overline{a}$ given by the smallest solution to equation (14). The reason is that if such an equilibrium existed, the welfare that a consumer would obtain in such a market from owning a car of age a is $V_{\overline{a}}(a)$ but by the Corollary this equals $W_{\overline{a}}(a) < W(a)$ where the latter is the *welfare a consumer can secure in autarky.* That is, $W(a)$ is the welfare a consumer can obtain from following a "buy and hold" trading strategy and never trading in the secondary market. Thus, we conclude that all consumers would abandon the secondary market if they had the freedom to choose when to scrap their autos. Thus, only the Pareto dominant equilibrium will exist if there are no legal restrictions on when consumers can scrap their autos. In the Pareto-dominant equilibrium consumers are indifferent between avoiding the secondary market and following a buy and hold strategy, or a frequent trading strategy of trading every period for a preferred car of age a^*. Since they are indifferent between holding any car of age a traded in the market, whether the secondary market exists or not is of no consequence to them. Thus, in order to provide an adequate explanation of why secondary markets for cars exist, we need to extend our model to allow for consumer heterogeneity which will provide a source of gains from trade.

3.2 Heterogeneous consumer economy with transactions costs

Similar to the homogeneous consumer model already presented, the heterogeneous consumer model is based on an economy with a continuum of consumers but where cars are indexed by a finite number of possible ages a where initially we assume there is a single type of car traded in the market. Thus, $a = 0$ indexes a brand new car and we continue to maintain the assumption that the government imposes a legal restriction that prevents consumers from owning cars after they reach an age $\overline{a}$ where they are considered no longer safe to drive and have only scrap value. We continue to make the "small open economy" assumption that there is an exogenously specified infinitely elastic supply of new cars at price $\overline{P}$ and an infinitely elastic demand for cars for their scrap value at price $\underline{P}$ where $\underline{P} < \overline{P}$.

We extend the model in the previous subsection in four ways. First, we allow an "outside good" i.e. a consumer's option not to own a car. To handle this we let $a = \emptyset$ denote the *state* of a consumer who currently owns no car and let $d = \emptyset$ denote the *decision* of a consumer not to own a car. Second, we allow for idiosyncratic time-varying heterogeneity indexed by continuously distributed multivariate extreme value random variables ε to be discussed in more detail below. Third, we allow for transactions costs and fourth, we allow for stochastic accidents that can lead to "premature scrapping" of automobiles. We will discuss the fourth extension in the next subsection, 3.3, and then we return to describe the rest of the model including the consumer value functions and our definition of equilibrium.

3.3 Stochastic Accidents

In the model discussed so far, the only way cars are scrapped (and thus removed from the economy) is when they reach the threshold age $\overline{a}$ where all cars are scrapped. Without stochastic accidents there will be a uniform invariant age distribution of cars on the set $\{1,2,\ldots,\overline{a}\}$ but this is not what we observe in the data: the age density of cars slopes downward so there are fewer old cars relative to new ones. In this section we extend the model to allow for stochastic accidents that result in downward sloping age distributions similar to what we observe.[8]

Let $\alpha(a)$ be the probability that a car of age a is involved in an accident that results in a total loss and its subsequent scappage and removal from the car stock. We represent this as a probability $\alpha(a)$ of transiting to the scrappage age $\overline{a}$ next period, where it is scrapped with probability 1. Thus, with probability $1-\alpha(a)$ a car of age a becomes age $a+1$ next period, and with probability $\alpha(a)$ it transits to the scrappage state $\overline{a}$. If a car is at the scrappage state $\overline{a}$, then if the economy is in 'flow equilibrium," the number of cars that are scrapped equal the number of new cars sold, since otherwise the stock of cars will either decrease or increase over time if scrappage of cars does not equal the production of new cars.

We can represent the aging of cars in the presence of stochastic age-dependent accidents via an $\overline{a}+1\times\overline{a}+1$ transition probability matrix Q given by

$$Q=\begin{bmatrix} 0 & 1-\alpha(0) & 0 & \cdots & 0 & \alpha(0) \\ 0 & 0 & 1-\alpha(1) & \cdots & 0 & \alpha(1) \\ 0 & 0 & \cdots & 1-\alpha(j) & \cdots & \alpha(j) \\ 0 & 0 & \cdots & 0 & 1-\alpha(\overline{a}-2) & \alpha(\overline{a}-2) \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1-\alpha(0) & 0 & \cdots & 0 & \alpha(0) \end{bmatrix}. \tag{28}$$

The rows of Q are indexed by the choice of car d at the start of the period, so $d=(0,1,\ldots,\overline{a})$ and the columns are indexed by the end of period age a of the car, where we represent an accident as a transition from any age $a<\overline{a}$ to the scrappage age $\overline{a}$ where we assume is an age that corresponds to such poor physical condition that the only releastic option is to scap the car.

To see how this transition probability matrix represents car aging and the impact of stochastic accidents, note for example the first row of Q which indicates that a brand new car will

[8]Though we focus on accidents for simplicity, it is possible to extend the analysis to allow for a variety of types of "stochastic deterioration" such as the case where the state of the car is captured by its odometer value (similar to the model of Rust (1985c)) and the stochastic state transitions can be interpreted as stochastic variability in usage/driving of cars by different consumers.

become a 1 year old car next period with probability $1-\alpha(0)$ (i.e. if there is no accident) or a car of age $\overline{a}$ with probability $\alpha(0)$. Note that a car of age $\overline{a}-1$ will become a car of age $\overline{a}$ with probability 1, regardless whether it experiences an accident or not. The last row of Q reflects the concept of *flow equilibrium* which is a restriction that once any car reaches the scrappage age $\overline{a}$ it is immediately scrapped and replaced by a new car. However we assume that the act of scrapping a car of age $\overline{a}$ and replacing it with a brand new on takes place instantaneously, so the transition probabilities in the first and last rows of Q are identical: a scrapped car is replaced by a brand new car, which either becomes a 1 year old car with probability $1-\alpha(0)$ if it has no accident, or it becomes a scrap vehicle (of age $\overline{a}$) if it has an accident, which occurs with probability $\alpha(0)$.[9]

The stationary age distribution of car holdings implied by the transition probablity matrix Q is a $(1\times\overline{a}+1)$ row vector q satisfying

$$q = qQ \tag{29}$$

We can see immediately that $q(0)=0$, i.e. our assumption that vehicles that are scrapped are instantaneously replaced by brand new vehicles implies that in steady state, we will not observe any brand new vehicles in the economy at the start of any period. In effect, brand new cars are only brand new for an instant, and thus occupy no actual mass in the stationary age distribution of vehicles. However there is a positive probability mass on vehicles of age $\overline{a}$, but since these vehicles are instantaneously replaced by brand new vehicles, we can interpret $q(\overline{a})$ as also equal to $q(0)$, though just for an instant. Since the market is in flow equilibrum, the number of cars that are scrapped equals the sales of brand new vehicles, so we can interpret $q(0)=q(\overline{a})$ as the volume of sales of new vehicles.

The other components of q give the invariant probabilities for ages $a=1,\dots,\overline{a}$ given by

$$q(\overline{a}) = \frac{1}{1+\sum_{a=1}^{\overline{a}-1}\prod_{i=0}^{a-1}(1-\alpha(i))} \tag{30}$$

and

$$q(a) = q(\overline{a})\prod_{i=0}^{a-1}(1-\alpha(i)) \quad a=1,\dots,\overline{a}-1. \tag{31}$$

If we make an alternative assumption that there is a one period lag between when a vehicle is scrapped and when it is replaced by a brand new vehicle, then the first element of the last row of

[9] An alternative treatment is to assume that a car of age $a=\overline{a}$ becomes a new car (age $a=0$) with probability 1. This amounts to assuming a one period lag between the time a car of age $\overline{a}$ is scrapped and when it is replaced by a new one.

Q equals 1, and then it is easy to see that if q is a solution to equation (29), then $q(0) = q(\overline{a}) > 0$. In this case there will be brand new vehicles in the steady state age distribution, but we can interpret them as "inventory" of brand new cars waiting to be sold and similarly $q(\overline{a})$ is a positive inventory of cars waiting to be scrapped. It is not hard to show that under this latter interpretation, the fraction of new cars in steady state will equal

$$q(0) = q(\overline{a}) = \frac{1}{2 + \sum_{a=1}^{\overline{a}-1} \prod_{i=0}^{a-1} (1 - \alpha(i))} \tag{32}$$

but the fractions of other ages of vehicles continues to be given by formula (31). In the remainder of this paper we adopt the original interpretation of instantaneous replacement of scrapped vehicles by brand new ones. Under this interpretation there is never any "inventory" of brand new vehicles so $q(0) = 0$, but in steady state flow equilibrium requires that the sales of new vehicles equals $q(\overline{a})$, the fraction of vehicles of the oldest age $\overline{a}$ that are scrapped.

Since there will be no cars of age 0 in the invariant holdings distribution at the start of any time t, it is convenient to work with the reduced $\overline{a} \times \overline{a}$ transition probabiity matrix, given by the lower $\overline{a} \times \overline{a}$ submatrix of Q in equation (28) which we will henceforth also refer to as Q

$$Q = \begin{bmatrix} 0 & 1-\alpha(1) & \cdots & \alpha(1) \\ 0 & \cdots & \cdots & \cdots \\ \cdots & 0 & 1-\alpha(\overline{a}-2) & \alpha(\overline{a}-2) \\ 0 & \cdots & 0 & 1 \\ 1-\alpha(0) & \cdots & 0 & \alpha(0) \end{bmatrix}. \tag{33}$$

It is easy to see that the last $\overline{a}$ components of the invariant probability q from equation (28) is also an invariant probability distribution for the transition submatrix Q given in equation (33).

Similar to the homogeneous consumer case, we assume that there are legal restrictions that prevent consumers from driving/owning cars that are older than the threshold age $\overline{a}$ where the resale price of a car equals the value, $P(\overline{a}) = \underline{P}$, and will therefore be scrapped. The technical reason why we make this assumption will become clearer shortly when we define equilibrium, but it has to deal with the difficulty of mathematically representing the infinitely elastic demand for vehicles at the exogenously specified scrap value $\underline{P}$.

3.4 Consumer value functions and definition of equilibrium

We assume that there are a continuum of consumers in the economy and that all time varying heterogeneity is idiosyncratic, that is, independently distributed across different consumers. In this paper we do not allow for any "macro shocks" that are experienced by all consumers. An example would be a time-varying fuel price that affects the utility of driving for all consumers, or time variation in the price of new cars that all consumers must pay. Thus we will be focusing on a *stationary equilibrium* similar to the one defined in Rust (1985c) where there is a time invariant price function $P(a)$ satisfying $P(0) = \overline{P}$, $P(a) = \underline{P}$ for $a \geq \overline{a}$, and for each $a \in \{1,\ldots,\overline{a}-1\}$ $P(a)$ equates the supply of cars of age a sold by existing consumers in the secondary market to the demand for cars of age a by other consumers in the economy. We assume there is an infinitely elastic supply of new cars at price $\overline{P}$ which implies that $P(0) = \overline{P}$, and we also assume there is an infinitely elastic demand for cars for their scrap value $\underline{P}$ which implies that $P(\overline{a}) = \underline{P}$, where $\overline{a}$ is the smallest age where $P(a)$ equals $\underline{P}$.

The assumption of an infinitely elastic demand for cars for their scrap value, regardless of what age that they are scrapped, implies that $P(a) \geq \underline{P}$. However the scrappage age $\overline{a}$ should be endogenously determined in equilibrium, similar to the way the scrappage threshold γ was endogenously determined in the stationary equilibrium analyzed in Rust (1985c), and the finite state case in section 3.1. Unfortunately it is not clear how to do this in a heterogeneous consumer economy with transactions costs because we are not aware of a way of representing equilibrium prices as shadow prices to a social planning problem. We now describe an alternative approach to the endogenous determination of $\overline{a}$ by iteratively searching for a "maximal equilibrium" similar to the one defined in section 3.1 except that in the heterogenous consumer case the endogenously determined value of $\overline{a}$ in the maximal equilibrium does not have an interpretation as the optimal stopping threshold to a social planning problem.

We continue to impose a restriction that consumers are not allowed to own cars that are equal to or older than the scrappage age $\overline{a}$. This is in part empirically motivated (i.e. most governments inspect vehicles and do not allow them to be driven if they are sufficiently old or unsafe), but we acknowledge that there is an artificial aspect of these restriction because it fails to specify how the government determines the appropriate value for $\overline{a}$. Similar to the homogeneous consumer case analyzed in section 3.1, we will show there can be multiple Pareto-ranked equilibria for different values of $\overline{a}$ provided $\overline{a}$ is not too large. However when $\overline{a}$ is sufficiently large, it will not be possible to find values of prices P that set excess demand to zero and satisfy the restriction $P(a) \geq \underline{P}$ for $a \in \{1,\ldots,\overline{a}-1\}$. This is because consumers prefer newer cars to older cars so it will not be possible to induce sufficiently many consumers to buy old cars of age a close to the scrappage age $\overline{a}$ unless $P(a)$ is set below $\underline{P}$ when $\overline{a}$ is sufficiently large. Since such candidate solutions violate the equilibrium constraint that $P(a) \geq \underline{P}$ implied by our assumption of an infinitely elastic demand for cars for their scrap value, it follows that there will be an upper bound on the value of $\overline{a}$.

Definition: *A maximal equilibrium is the largest value of the scrappage age $\overline{a}$ for which there exists a solution to the system of equations $ED(P) = 0$, where $ED : R^{\overline{a}-1} \to R^{\overline{a}-1}$ is a system of equations defining the excess demands by consumers for cars of ages $a \in \{1,2,\ldots,\overline{a}-1]$ (to be defined below) that satisfies the restriction that $P(a) \geq \underline{P}$ for $a \in \{1,2,\ldots,\overline{a}-1\}$.*

We assume that a maximal equilibrium exists and provide an algorithm for computing it. Our algorithm uses Newton's method to solve the system of equations $ED(P) = 0$ for sequence of candidate values of $\overline{a}$. At the i^{th} step of this iterative process the candidate equilibrium has scrappage age $\overline{a}_i$ and a price vector P_i. We check the solution to see if it satisfies the constraint $P_i(a) \geq \underline{P}$ for $a \in \{1,2,\ldots,\overline{a}_i - 1\}$. If so, then we increment $\overline{a}_i$ by one and search for a new equilibrium until we encounter a value of $\overline{a}_i$ where these price constraints are violated. The maximal equilibrium is given by scrappage age $\overline{a}_{i-1}$ from the previous iteration and the equilibrium prices are given by the vector P_{i-1}.

We argue that maximal equilibrium is a reasonable solution concept and equilibrium selection rule for the car market. We can interpret it as the outcome of a decentralized trading process where the used price $P(a)$ is bid up whenever there is an excess demand for a car of age a, and conversely $P(a)$ falls whenever there is an excess supply for cars of age a. Once a decentralized market equilibrium obtains, the government codifies it by imposing restriction that cars are no longer safe to drive and must be scrapped when they reach age $\overline{a}$. Under this interpretation, the only real constraint imposed by the government restriction is that it prevents consumers from buying and holding cars in excess of the scrappage age $\overline{a}$ for their "collector's value" – a phenomenon that is difficult to capture in our model.[10]

We introduce a flexible specification for transactions cost via a function $T(P,d,a)$ which depends on the full vector of new and used car prices and the age d of a car a consumer wants to buy and potentially the age a of car that the consumer sells. An example that falls within this

[10] An alternative approach to computing equilibrium would be to relax the assumption that consumers are unable to own vehicles beyond a government-defined scrappage age $\overline{a}$ and allow individual consumers to determine the time when prefer to scrap their vehicles, possibly allowing additional heterogeneity in the form of extreme value shocks that affect the binary decision of whether to sell a car in the used market or scrap it similar to the specifications of Manski and Goldin (1983) and Berkovec (1985). However any algorithm for computing an equilibrium will need to impose some bound on the set of possible vehicle ages that can be traded or held in equilibrium. Thus we do not see any way to avoid imposing an arbitrary restriction that consumers are not allowed to keep cars that are sufficiently old.

general specification is given by

$$T(P,d,a) = T + \rho[P(d) - P(a)] \tag{34}$$

where $\rho \geq 0$ is a proportional transactions costs or sales tax on the net cost of the vehicle trade, $P(d) - P(a)$. We assume that transactions costs are borne by the buyer. There is an element of arbitrariness whether transactions costs are born by the seller or buyer, since inspection and repair costs incurred by a seller (such as a car dealer) could be added to the price the dealer charges and thus be equally regarded as paid by the buyer. We adopt the convention that these costs are paid by the buyer and this includes any sales or new car taxes charged by the government.

We assume that consumers live forever and discount future utility at a common value $\beta \in (0,1)$. The *state* of a consumer is given by the vector (a,ε) where a denotes the age of the current car that the consumer owns ($a \in \{1,2,\ldots,\overline{a}\}$) or $a = \emptyset$ if the consumer does not currently own a car. Similar to section 3.1 we assume that consumers want at most one car so we assume there is zero marginal utility from owning more than one car. ε is a vector with the same number of components as the number of choices the consumer has, and represents the idiosyncratic, time-varying extreme valued idiosyncratic shocks that capture a variety of phenomena such as search costs, maintenance expenditures, and deviations in car transaction prices from the "blue book" or expected prices to the realized values in particular transactions that reflect specific features of the cars being traded that the buy and seller can observe but which are not observed by the econometrician. Thus, ε_t will be an *IID* shock process that is independent over time for a particular consumer and independently distributed across consumers in the economy. The state variable a_t will be serially correlated as a result of the consumer's vehicle holding and trading decisions, but we also assume this state is independently distributed across different consumers.

Let $V(a,\varepsilon)$ be the value function for a consumer in state (a,ε), and let $D(a)$ be the set of feasible choices for a consumer whose car state is a. If the consumer has no car, $a = \emptyset$, they can remain in the no-car state or choose one of the ages of cars available for sale in the market, so we have

$$D(\emptyset) = \{\emptyset, 0, 1, \ldots, \overline{a} - 1\}, \tag{35}$$

so the decision $d = \emptyset$ corresponds to staying in the no-car state, $d = 0$ corresponds to buying a new car, and so forth. If the consumer owns a car of age a, we let the choice $d = -1$ denote the decision to keep the current car rather than trade it for another vehicle. So the choice set in this case is given by

$$D(a) = \{\emptyset, -1, 0, 1, \ldots, \overline{a}-1\}. \tag{36}$$

Note the distinction between the decision $d = -1$ (keep the current car of age a) and $d = a$ (trade for another car of the same age a). These are not equivalent choices since the latter involves search and transactions costs that are avoided by the decision to keep the current vehicle.

Consider the Bellman equation for $V(\emptyset, \varepsilon)$, the discounted utility of a consumer who does not own a car.

$$V(\emptyset, \varepsilon) = \max\left[v(\emptyset, \emptyset) + \varepsilon(\emptyset), \max_{d \in \{0,1,\ldots,\overline{a}-1\}} [v(d, \emptyset) + \varepsilon(d)]\right], \tag{37}$$

where

$$\begin{aligned} v(d, \emptyset) &= u(d) - \mu[P(d) + T(P, d, \emptyset)] + \beta\left[(1-\alpha(d))EV(d+1) + \alpha(d)EV(\overline{a})\right] \\ v(\emptyset, \emptyset) &= u(\emptyset) + \beta EV(\emptyset), \end{aligned} \tag{38}$$

where $u(d)$ is the utility of a d-year old car, $u(\emptyset)$ is the utility of the no-car state, and $EV(a)$ and $EV(\emptyset)$ are the conditional expectations of the value functions $V(a, \varepsilon)$ and $V(\emptyset, \varepsilon)$, which represents the expectation of future utility for a consumer who owns a car of age a and who does not currently own a car, respectively. Similarly, the Bellman equation for a consumer who owns a car of age a has both the option to "purge" their car, i.e. sell the car but not buy another to replace it, and to keep their car (as given by the decision $d = -1$) in addition to trading it for another car. So the value function is given by

$$V(a, \varepsilon) = \max\left[v(\emptyset, a) + \varepsilon(\emptyset), v(-1, a) + \varepsilon(-1), \max_{d \in \{0,1,\ldots,\overline{a}-1\}} [v(d, a) + \varepsilon(d)]\right], \tag{39}$$

where $v(\emptyset, a)$ is the value of selling one's current car of age a and not replacing it

$$v(\emptyset, a) = u(\emptyset) + \mu P(a) + \beta EV(\emptyset), \tag{40}$$

and $v(-1, a)$ is the value of keeping the current car of age a

$$v(-1, a) = u(a) + \beta\left[(1-\alpha(a))EV(a+1) + \alpha(a)EV(\overline{a})\right], \tag{41}$$

and $v(d,a)$ is the value of trading the current car a for a car of age $d \in \{0,1,\ldots,\overline{a}-1\}$

$$v(d,a) = u(d) - \mu[P(d) - P(a) + T(P,a,d)] + \beta\left[(1-\alpha(d))EV(d+1) + \alpha(d)EV(\overline{a})\right]. \quad (42)$$

When ε is a vector of *IID* preference shocks with a Type 1 extreme value distribution with mean value normalized to zero and scale parameter σ, we can use the "max-stable" property (i.e. that the family of extreme value distributions are closed under the max operator, see e.g. McFadden (1981)) to write

$$EV(\emptyset) \equiv \int V(\emptyset,\varepsilon)dF(\varepsilon) = \sigma \log\left(\exp\{v(\emptyset,\emptyset)/\sigma\} + \sum_{d=0}^{\overline{a}-1} \exp\{v(d,\emptyset)/\sigma\}\right), \quad (43)$$

where $F(\varepsilon)$ is the CDF of the multivariate Type 1 extreme value distribution. Similarly we have

$$EV(a) = \sigma \log\left(\exp\{v(\emptyset,a)/\sigma\} + \exp\{v(-1,a)/\sigma\} + \sum_{d=0}^{\overline{a}-1} \exp\{v(d,a)/\sigma\}\right) \quad (44)$$

for $a \in \{1,2,\ldots,\overline{a}-1\}$, and

$$EV(\overline{a}) = \sigma \log\left(\exp\{v(\emptyset,a)/\sigma\} + \sum_{d=0}^{\overline{a}-1} \exp\{v(d,a)/\sigma\}\right) \quad (45)$$

where the last equation reflects the constraint that consumers are not allowed to keep their car once it reaches the scrappage age $\overline{a}$. We can stack equations (43) to (45) above into a single system of nonlinear equations for the $(\overline{a}+1) \times 1$ vector of expected value functions $EV = (EV(\emptyset), EV(1), \ldots, EV(\overline{a}))$ as

$$EV = \Gamma(EV,P) \quad (46)$$

where Γ is the "smoothed Bellman operator" see, e.g. Rust (1985b).

Lemma 0 Γ *is a contraction mapping in EV and hence has a unique fixed point $EV(P)$ for each value of P.*

Proof: This follows from Blackwell's sufficient condition for a contraction mapping, which is implied by the monotonicity and the quasilinearity properties of the Γ operator, i.e. 1) if $EV' \geq EV$ then $\Gamma(EV',P) \geq \Gamma(EV,P)$, and 2) if e is a vector of ones of dimension $\overline{a}+1 \times 1$ and α is any positive scalar, then $\Gamma(EV+\alpha e,P) = \Gamma(EV,P) + \beta\alpha$, and the fact that $\beta \in (0,1)$, see Rust, Traub and Wozniakowski (2002).

The consumer's dynamic programming can be solved in the usual way, by finding a fixed point $V = \Lambda(V)$ of the standard (non-smoothed) Bellman equation, however it is much faster computationally to solve the problem by first finding the unique fixed point EV to the smoothed Bellman operator Γ in equation (46) and then constructing successively, the "decision-specific" value functions $v(d,a)$ given in equations (40), (41), and (42) above, then then in turn using these to construct the usual value function $V(a,\varepsilon)$ given in equation (39). The reason, of course, is to avoid the curse of dimensionality. Consider a case where $\overline{a} = 50$. Then solving for EV from the smoothed Bellman operator in equation (46) involves computing a contraction fixed point in R^{51}, whereas solving a related version of the Bellman equation for the decision-specific value functions $v(d,a)$ would require solving for a fixed point in R^{2651} and directly solving the Bellman equation (39) requires approximating the fixed point $V(a,\varepsilon)$ to a contraction mapping on an infinite-dimensional space of functions of up to 52 continuous state variables, ε.

We obtain further computational speed-up by using the method of *policy iteration* (Howard (1960)) which is mathematically equivalent to the use of Newton's method to solve for EV as a zero of the mapping $H(EV,P) = EV - \Gamma(EV,P) = 0$. It is easy to see from the definition of the smoothed Bellman operator Γ that this is a continuously differentiable function of EV so its gradient $\nabla_{EV}\Gamma(EV,P)$ exists and is a continuous function of EV for any P. Then the policy iteration (Newton's method) algorithm provides a sequence of approximations $\{EV_t\}$ to the zero $H(EV,P) = 0$ given by

$$EV_{t+1} = EV_t - [I - \nabla_{EV}\Gamma(EV_t,P)]^{-1}[EV_t - \Gamma(EV_t,P)]. \tag{47}$$

Policy iteration typically converges after a small number of iterations (typically fewer than 10) to a highly accurate approximation of the true fixed point. It is typically far faster and more accurate than the use of sucessive approximations $EV_{t+1} = \Gamma(EV_t,P)$ that is the standard method for approximating contraction fixed points. To find an equilibrium we will be repeatedly invoking Newton's method for an "outer iteration" to find a zero (in prices P) of the excess demand function $ED(P) = 0$, but for each value of P from this outer algorithm we need to find the corresponding fixed point $EV = \Gamma(EV,P)$ to the smoothed Bellman operator. So this "inner" fixed point will be repeatedly calculated and thus it is important to use the fastest and most accurate algorithms possible to solve the consumer's dynamic programming problem.

In order to define the excess demand function, we need to have formulas for the conditional choice probabilites $\Pi(d|a,P)$, which represents the probability that a consumer who is in state a will choose option $d \in D(a)$. Under the extreme value assumption for $F(\varepsilon)$ it is well known that $\Pi(d|a,P)$ is given by the standard multinomial logit formula. When $a < \overline{a}$ we have

$$\Pi(d|a,P) = \frac{\exp\{v(d,a,P)/\sigma\}}{\exp\{v(d,a,P)/\sigma\} + \exp\{v(-1,a,P)/\sigma\} + \sum_{d'=0}^{\overline{a}-1} \exp\{v(d',a,P)/\sigma\}}, \tag{48}$$

and when $a = \overline{a}$ the choice probability is

$$\Pi(d|\overline{a},P) = \frac{\exp\{v(d,\overline{a},P)/\sigma\}}{\exp\{v(\emptyset,\overline{a},P)/\sigma\} + \sum_{d=0}^{\overline{a}-1} \exp\{v(d,\overline{a},P)/\sigma\}}, \tag{49}$$

reflecting the constraint that the consumer cannot keep a car of age $\overline{a}$ or older, and if $a = \emptyset$ the choice probability is given by

$$\Pi(d|\emptyset,P) = \frac{\exp\{v(d,\emptyset,P)/\sigma\}}{\exp\{v(\emptyset,\emptyset,P)/\sigma\} + \sum_{d=0}^{\overline{a}-1} \exp\{v(d,\emptyset,P)/\sigma\}}. \tag{50}$$

There are only two other objects we need to introduce in order to define the excess demand function $ED(P)$ and equilibrium in this model. The trading dynamics implied by the model are given by the $(\overline{a}+1 \times \overline{a}+1)$ Markov transition probability matrix $\Delta(P)$ given by

$$\Delta(P) = \begin{bmatrix} \Pi(1|1)+\Pi(-1|1) & \Pi(1|2) & \cdots & \Pi(\overline{a}-1|1) & \Pi(0|1) & \Pi(\emptyset|1) \\ \Pi(1|2) & \Pi(2|2)+\Pi(-1|2) & \cdots & \Pi(\overline{a}-1|2) & \Pi(0|2) & \Pi(\emptyset|2) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \Pi(1|\overline{a}) & \Pi(2|\overline{a}) & \cdots & \Pi(\overline{a}-1|\overline{a}) & \Pi(0|\overline{a}) & \Pi(\emptyset|\overline{a}) \\ \Pi(1|\emptyset) & \Pi(2|\emptyset) & \cdots & \Pi(\overline{a}-1|\emptyset) & \Pi(0|\emptyset) & \Pi(\emptyset|\emptyset) \end{bmatrix}, \tag{51}$$

where we have suppressed the dependence of the choice probabilities on the price vector P to simplify notation. Notice that the diagonal elements of $\Delta(P)$ equal $\Pi(a|a)+\Pi(-1|a)$, i.e. the probability of remaining in the same state a equals the sum of the probability that the consumer chooses to trade their vehicle of age a for another vehicle of age a, $\Pi(a|a)$, plus the probability that the consumer chooses not to trade and keep their current vehicle of age a, $\Pi(-1|a)$. However we do not allow a consumer whose car is of age $\overline{a}$ to keep this vehicle, or buy a vehicle of this age, so we restrict $\Pi(-1|\overline{a}) = 0$ and $\Pi(\overline{a}|a) = 0$ for $a \in \{1,2,\ldots,\overline{a}-1\}$.

We refer to $\Delta(P)$ as the *trade transition probability matrix* because it represents the dynamics of how consumers change the status of owning or not owning cars during the instantaneous trades that take place just after the start of each period t but before the economy transits to the next period $t+1$. Notice the subtle notational distinction between the empty set symbol $\emptyset$ and the numeral 0: for example $\Pi(\emptyset|\emptyset)$ represents the "self-transition" from the no car state to itself

— i.e. the probability that a consumer who does not own a car at the start of period t chooses to remain without a car (i.e. to choose the "outside good" again), whereas $\Pi(0|\emptyset)$ represents the fraction of consumers who do not have a car at the start of period t who choose to purchase a brand new car, $d = 0$. Clearly the transition probabilities in the first row of $\Delta(p)$ must add up to 1, since all consumers who do not own a car face the same choice set $d \in \{\emptyset, 0, 1, \ldots, \overline{a}-1\}$.

Let Ω be the $(\overline{a}+1) \times (\overline{a}+1)$ Markov transition matrix formed by bordering the $\overline{a} \times \overline{a}$ accident/aging transition probability matrix Q for cars given in equation (33) as follows

$$\Omega = \begin{bmatrix} Q & 0 \\ 0' & 1 \end{bmatrix} \tag{52}$$

where 0 is an $\overline{a} \times 1$ vector of zeros. We continue to refer to Ω as the *aging/accident transition probability matrix* because it summarizes all possible transitions in car holdings including individuals who do not have a car. The latter transition probability is captured by the constant 1 in the $(\overline{a}+1, \overline{a}+1)$ element of this matrix and reflects the natural constraint that a consumer who does not own a car at the start of the period will also not own a car at the end of the period.

The first lemma below establishes that EV, the value functions v, and the choice probabilities Π, and thus the full transition probability matrix $\Delta(P)$ are continuously differentiable functions of prices, P. This lemma is a key result for the applicability of Newton's method, since the excess demand function $ED(P)$ will be defined in terms of choice probabilities and an invariant probability measure $q(P)$ that we define below. In order to show that $ED(P)$ is a continuously differentiable function of P we need to first show that all of these other objects are continuously differentiable in P.

Lemma 1 *Let $EV = \Gamma(EV, P)$ be the unique fixed point of the smoothed Bellman operator Γ and let $\nabla_{EV}\Gamma(EV, P)$ be the gradient of Γ with respect to EV. Then we have:*

$$\nabla_{EV}\Gamma(EV, P) = \beta\Delta(P)\Omega \tag{53}$$

where $\Delta(P)$ is the transition probability matrix given in equation (51) and Ω is the transition probability matrix given in equation (52).

Proof: This can be shown via algebra after differentiating the Γ operator with respect to EV in equations (43), (44) and (45) above and expressing the result as a matrix product. □

Lemma 2 *EV, the unique fixed point of the smoothed Bellman operator $EV = \Gamma(EV, P)$ in equation (46), the choice-specific value functions v, the choice probabilities Π, and trade transition probability matrix $\Delta(P)$ are continuously differentiable functions of P.*

Proof: This follows from the Implicit Function Theorem provided we can show that 1) Γ is a continuously differentiable function of P, and 2) the $(\overline{a}+1)\times(\overline{a}+1)$ matrix $I-\nabla_{EV}\Gamma(EV,P)$ is nonsingular for any P where $EV=\Gamma(EV,P)$, where ∇_{EV} denotes the gradient operator with respect to EV. However by Lemma 1 we have $\nabla_{EV}\Gamma(EV,P)=\beta\Delta(P)\Omega$. This implies that $I-\nabla_{EV}\Gamma(EV,P)$ is nonsingular with inverse equal to

$$[I-\beta\Delta(P)\Omega]^{-1}=\sum_{t=0}^{\infty}\beta^t[\Delta(P)\Omega]^t \tag{54}$$

where the latter series is convergent due to the assumption that $\beta\in(0,1)$ and $\|\Delta(P)\Omega\|=1$. It is easy to show that $\nabla_P\Gamma(EV,P)$ exists and is continuous in EV and P using the formulas for Γ in equations (43), (44) and (45) above. Thus, by the Implicit Function Theorem we have that $EV(P)$ is a continuously differentiable function of P with gradient given by

$$\nabla_P EV(P)=[I-\beta\Delta(P)\Omega]^{-1}\nabla_P\Gamma(EV,P). \tag{55}$$

$\square$

Consider the $1\times(\overline{a}+1)$ row vector $q(P)$ defined by

$$q(P)=q(P)\Delta(P)\Omega \tag{56}$$

It is easy to see that equation (56) characterizes the *steady state or invariant holdings distribution of cars in the economy.* To see this, note that the product of the two $(\overline{a}+1)\times(\overline{a}+1)$ Markov transition probabilities $\Delta(P)$ and Ω is another Markov transition probability matrix that represents the *evolution of the age distribution of cars in the economy.* That is, at the start of period t assume there is an initial distribution $q(P)$ of car holdings in the economy, where the first component of $q(P)$ is the fraction of consumers who own cars aged 1 year old, and component $\overline{a}$ is the fraction of consumers who own cars aged $\overline{a}$ and the final component is the fraction of consumers who do not own a car. Then the product $q(P)\Delta(P)$ represents the distribution of holdings immediately after trading occurs at period t, which we have assumed occurs instantaneously. Thus, $q(P)\Delta(P)$ can be called the *post-trade holdings distribution* and it will have positive mass on consumers who own brand new vehicles but zero mass of individuals who own cars of age $\overline{a}$ due to the restriction that consumers are not allowed to keep cars once they reach the scrappage age $\overline{a}$. Notice that component $\overline{a}$ of the row vector $q(P)\Delta(P)$ equals the fraction of consumers who purchase *new cars.* It follows from equation (33) that the corresponding component of $q(P)\Delta(P)\Omega$ represents the fraction of consumers who hold cars of age $\overline{a}$. Thus, the first $\overline{a}$ components of $q(P)$ represent the holdings of cars of age $a = (1,2,\dots,\overline{a})$ and the last component represents the fraction of consumers who do not own a car.

Even though the transition matrix Q given in (33) has a unique invariant distribution $q = qQ$ (a row vector with $\overline{a}$ components), the transition matrix Ω in equation (52) has a continuum of invariant distributions corresponding to various possible divisions of consumers between owning cars and the outside good. Let $q(\emptyset) \in [0,1]$ be the fraction of consumers who have no car in steady state. We can express all invariant distributions to Ω as $(1 \times \overline{a}+1)$ vectors of the form $((1-q(\emptyset))q, q(\emptyset))$. However in a stationary equilibrium, the fraction of the consumers who choose the outside good will be uniquely determined as part of the equilibrium, and this implies that the transition matrix $\Delta(P)\Omega$ will have a unique invariant distribution $q(P)$, which defines the holdings distribution of cars as an implicit function of prices, P where the last component of this vector is $q(\emptyset,P)$, the fraction of consumers holding the outside good in equilibrium.

To see this, note that for any P, the last component of the matrix $q(P)\Delta(P)$ equals the fraction of the population who chooses to have no car. So in a stationary equilibrium, we must have

$$q(\emptyset,P) = q(\emptyset,P)\Pi(\emptyset|\emptyset,P) + [1-q(\emptyset,P)]\left[\sum_{a=1}^{\overline{a}} \Pi(\emptyset|a,P)q(a)\right]. \tag{57}$$

The left side of equation (57) is the “supply” of the outside good, i.e. the fraction of the population who did not own a car at the start of period t, and the right hand side of equation (57) is the “demand” for the outside good, i.e. the sum of the fraction of the popluation who do not own cars and choose not to buy a car again this period, plus the fraction of the population who do own cars, but who decide to sell them to “purchase the outside good” instead. In steady state, these two fractions must be the same, so equation (57) defines the fraction of consumers who choose to own no car in steady state, $q(\emptyset,P)$, as an implicit function of P. Thus, if the economy is in a stationary equilibrium, the fraction of the population choosing the outside good $q(\emptyset,P)$ is uniquely determined in this equilibrium, which in turn identifies the particular invariant distribution of the transition probability matrix Ω given by $q(P) = ((1-q(\emptyset,P))q, q(\emptyset,P))$.

In fact, we can solve equation (57) for this expression for $q(\emptyset,P)$ for any P resulting in

$$q(\emptyset,P) = \frac{\sum_{a=1}^{\overline{a}} \Pi(\emptyset|a,P)q(a)}{1-\Pi(\emptyset|\emptyset,P)+\sum_{a=1}^{\overline{a}} \Pi(\emptyset|a,P)q(a)}. \tag{58}$$

This formula has an intuitive interpretation: it is the invariant probability of the outside good state $\emptyset$ for a two state Markov chain with 2×2 transition probability matrix M given by

$$M = \begin{bmatrix} \Pi(\emptyset|\emptyset,P) & 1-\Pi(\emptyset|\emptyset,P) \\ \sum_{a=1}^{\overline{a}}\Pi(\emptyset|a,P)q(a) & 1-\sum_{a=1}^{\overline{a}}\Pi(\emptyset|a,P)q(a) \end{bmatrix}, \tag{59}$$

where $\Pi(\emptyset|\emptyset,P)$ is the transition probability for staying in the "outside good state" whereas the lower right $(2,2)$ element is the transition probability for staying in the "car state", and $q(\emptyset,P)$ is the invariant probability of the no-car state implied by the two state Markov chain M. Once we know this probability, we have the invariant probabilities for the various car ages are $[1-q(\emptyset,P)]q(a)$, $a=1,\dots,\overline{a}$ where $q(a)$ are given by formulas (30) and (31) above.

Now we are in position to define excess demand. First we note that there are $\overline{a}-1$ "unknowns", the prices $P=(P(1),\dots,P(\overline{a}-1))$. We define the excess demand function $ED(P)$: $R^{\overline{a}-1}\to R^{\overline{a}-1}$ as

$$ED(P) = D(P) - S(P) \tag{60}$$

where the demand function is given by

$$D(P) = \begin{bmatrix} \Pi(1|\emptyset,P)q(\emptyset,P)+[1-q(\emptyset,P)]\left[\sum_{a=1}^{\overline{a}}\Pi(1|a,P)q(a)\right] \\ \Pi(2|\emptyset,P)q(\emptyset,P)+[1-q(\emptyset,P)]\left[\sum_{a=1}^{\overline{a}}\Pi(2|a,P)q(a)\right] \\ \cdots \\ \Pi(\overline{a}-2|\emptyset,P)q(\emptyset,P)+[1-q(\emptyset,P)]\left[\sum_{a=1}^{\overline{a}}\Pi(\overline{a}-2|a,P)q(a)\right] \\ \Pi(\overline{a}-1|\emptyset,P)q(\emptyset,P)+[1-q(\emptyset,P)]\left[\sum_{a=1}^{\overline{a}}\Pi(\overline{a}-1|a,P)q(a)\right] \end{bmatrix} \tag{61}$$

and supply is given by

$$S(P) = \begin{bmatrix} [1-q(\emptyset,P)]q(1)[1-\Pi(-1|1,P)] \\ [1-q(\emptyset,P)]q(2)[1-\Pi(-1|2,P)] \\ \cdots \\ [1-q(\emptyset,P)]q(\overline{a}-2)[1-\Pi(-1|\overline{a}-2,P)] \\ [1-q(\emptyset,P)]q(\overline{a}-1)[1-\Pi(-1|\overline{a}-1,P)]. \end{bmatrix} \tag{62}$$

where $q(a)$, $a=1,\dots,\overline{a}$ is the invariant distribution of car ages given by formulas (30) and (31) above and $q(\emptyset,P)$ is the fraction of the population choosing the outside good given by equation (58).

Theorem 1 *Let P be a stationary equilibrium price vector with equilibrium scrappage threshold*

$\overline{a}$, i.e. a solution to $ED(P)=0$. Then the implied holdings distribution $q(P)$ is an invariant distribution of $\Delta(P)$, i.e.

$$q(P)=q(P)\Delta(P) \tag{63}$$

Proof In a stationary equilibrim equation (58) holds as discussed above. It is not hard to show that by rearranging this equation and the equilbrium condition $ED(P)=0$, and the fact that the components of $q(P)$ must add to 1 for any P that equation (63) must hold, i.e. $q(P)$ is an invariant distribution of $\Delta(P)$. □

Theorem 2 *Let P be a stationary equilibrium price vector with equilibrium scrappage threshold $\overline{a}$ from a solution to $ED(P)=0$ and let $\Delta(P)$ be trade transition probability matrix given in equation (51). Then the invariant probability distribution $q(P)$ given in (63) is the unique invariant probability distribution for the transition probability matrix Ω in equation (52) and thus also of the product matrix $\Delta(P)\Omega$*

$$q(P)=q(P)\Delta(P)=q(P)\Omega=q(P)\Delta(P)\Omega. \tag{64}$$

Proof We have already shown that the set of all possible invariant distributions of Ω takes the form $((1-q(\emptyset))q,q(\emptyset))$ for any $q(\emptyset)\in[0,1]$. Notice that the invariant distribution $q(P)=q(P)\Delta(P)$ in equation (63) above takes this form for the particular value $q(\emptyset)=q(\emptyset,P)$ where $q(\emptyset,P)$ is given in (58). So this implies that $q(P)=q(P)\Omega$, i.e. $q(P)$ is the only invariant distribution of Ω that is also consistent with equilibrium in the secondary market. The final equality in equation (64) follows from simple substitution and shows that $q(P)$ is also an invariant distribution of $\Delta(P)\Omega$. □

Theorems 1 and 2 serve as elegantly simple characterizations of stationary equilibrium prices and quantities (holdings distribution) for an exogenously specified scrappage age $\overline{a}$. In Theorem 3 we establish that for any $\overline{a}>1$ at least one stationary equilibrium will exist.

Theorem 3 *For any $\overline{a}>1$ there exists a stationary equilibrium price and quantity vector $(P,q(P))$ satisfying (64).*

Proof: For any given $\overline{a}$ define a mapping $\Lambda(P):R^{\overline{a}-1}\rightarrow R^{\overline{a}-1}$ by

$$\Lambda(P)=P+ED(P) \tag{65}$$

where R^n_+ denotes the positive orthant of R^n and $ED(P)$ is defined in equation (60) above. From the lemmas above, it follows that ED and thus Λ is a continuous mapping from $R^{\overline{a}-1}\rightarrow R^{\overline{a}-1}$. Note also that for any P the components of $ED(P)$ lie in the interval $[-1,1]$. Thus, when prices are sufficiently high, a vanishing number of consumers will wish to buy any new car but nearly all consumers will want to sell their cars, so $ED(P)$ will be close to a vector with all its components equal to -1. Similarly, for a sufficiently low set of prices (possibly negative), nearly all consumers will wish to buy used cars and very few will want to sell their vehicles at such low prices. So for such prices $ED(P)$ will be close to a vector with all of its components equal to $+1$. It follows that we can define a compact box B in $R^{\overline{a}-1}$ where Λ satisifies an "inward pointing" property on the boundaries of this box, so it follows that $\Lambda : B \to B$. Since Λ is a continuous mapping and B is a compact, convex set, the Brouwer fixed point theorem implies that a fixed point to Λ exists, and it is clear that any such fixed point is a stationary equilibrium, at least in the sense that it sets $ED(P) = 0$ and solves equation (64). $\square$

Theorem 3 implies a multiplicity of solutions $ED(P)$ indexed by different choices of the scrappage threshold $\overline{a}$. However not all such solutions are 'valid" stationary equilibria because as we noted, a valid equilibrium must also satisfy the stronger requirement that $P(a) \geq \underline{P}$, $a \in \{1,\ldots,\overline{a}-1\}$ that is implied by our assumption of an infinitely elastic demand for cars for their scrap value $\underline{P}$. We defined a *maximal equilibrium* as the largest value of $\overline{a}$ such that this additional restriction on prices is satisified. The existence of a maximal equilibrium can be established via additional assumptions on preferences: specifically, if the utility from a car $u(a)$ is strictly monotonically decreasing and unbounded below, then for sufficiently large values of $\overline{a}$ and values of a sufficiently close to $\overline{a}$, a negligible fraction of consumers will choose to buy such old cars unless their price falls below $\underline{P}$ or even takes negative values. Any such value of $\overline{a}$ will constitute an upper bound on the scrappage age in a maximal equilibrium. On the other hand, for values of $\overline{a}$ that are sufficiently small, it will generally be possible to find a stationary equilibrium that satisifes the constraint $P(a) \geq \underline{P}$ for $a \in \{1,2,\ldots,\underline{a}-1\}$. Any such value of $\overline{a}$ is a lower bound for the scrappage age in a maximal equilibrium. Thus, we can find a maximal equilibrium iteratively by starting with an initial guess for $\overline{a}$ and sequentially increasing it by 1 and computing the corresponding stationary equilibrium $(P,q(P))$ until we reach a point where P violates the inequality restrictions.

Rather than impose general restrictions on preferences to ensure the existence of a maximal stationary equilibrium, the virtue of our computational approach is that we can establish existence constructively, via a direct computation of the equilibrium using the iterative algorithm described above. Figure 1 illustrates a computed maximal equilibrium for example where $\overline{P} = 190$, $\underline{P} = 1$, $\beta = 0.95$, $\sigma = 1$ and $T(P,a,d) = 1 + .03P(d)$. We used the scrappage threshold and shadow prices from the social planning problem as our initial guess for $\overline{a}$ and P. In the maximal equilibrium $\overline{a} = 11$ whereas the initial guess from the solution to the homogenous consumer, no transactions cost social planning problem was $\overline{a} = 10$.

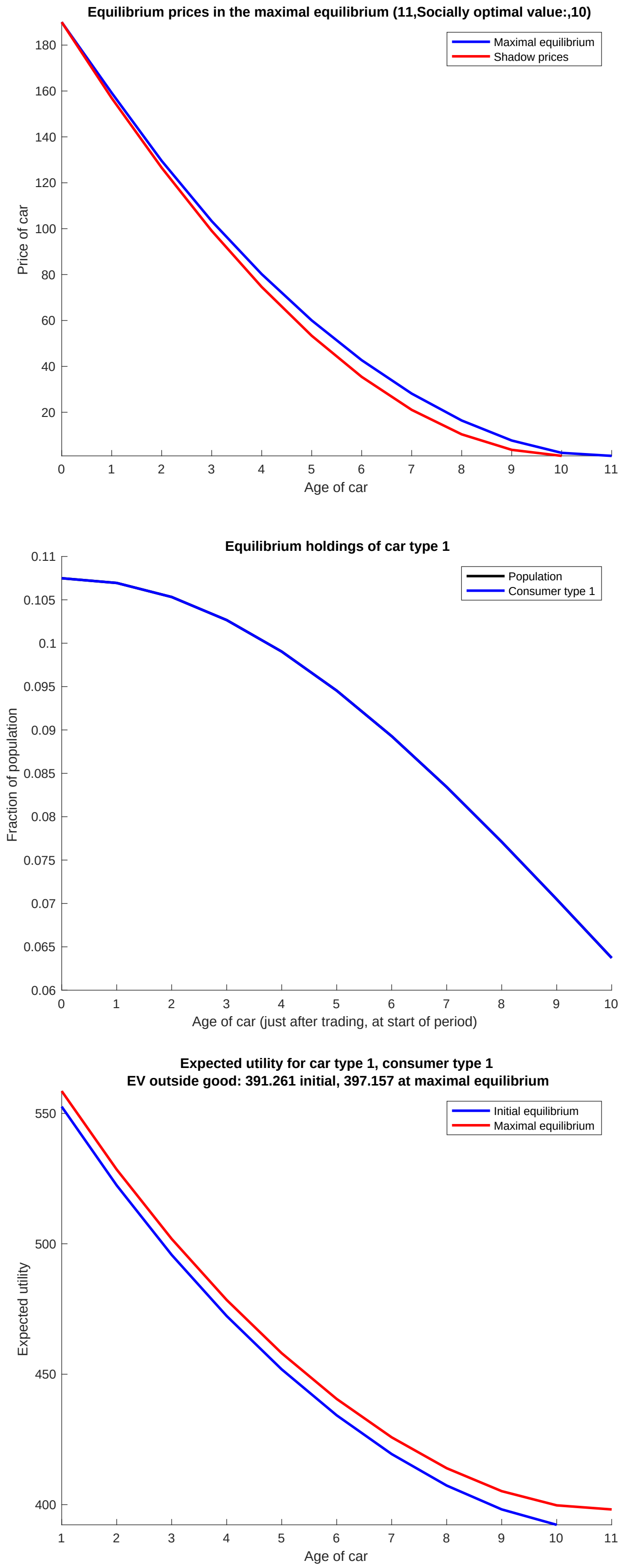
There are several points to notice from the example equilibrium in figure 1. First the homogeneous agent, no transaction cost theory presented in section 3.1 is definitely not irrelevant: we see from the top panel in this figure that the shadow prices to the social planning problem problem provide a very good initial guess for the prices in the heterogenous agent maximal equilibrium with transactions costs. The second panel presents the stationary holdings distribution $q(P)$ and we see that due to accidents, the age distribution of cars has a realistic downward sloping shape. Given our normalization of the utility of the outside good $u(\emptyset) = 0$, nearly the entire population chooses to own a car in steady state, and 10.75% of the population scraps old cars when then reach age $\overline{a} = 11$ or have an accident, so in flow equilibrium there is an equal fraction of consumers buying new cars (though not necessarily the same people who have an accident or scrap their old cars).

The final panel of figure 1 plots $EV(a)$ for $a \in \{1, \dots, \overline{a}\}$ to show the welfare gains obtained from increasing the initial guess of stationary equilibrium with $\overline{a} = 10$ to the maximal equilibrium where $\overline{a} = 11$. We see there is a strict Pareto improvement, and even the small fraction of individuals who do not have a car experience a Pareto gain, since the value of owning a car is higher in the maximal equilibrium. Thus, even though transactions costs inhibit trade, the gains from trade resulting from consumer heterogeneity (as captured by the inclusion of the idiosyncratic extreme value shocks) more than compensates for the dampening effect of transactions costs and cars actually live one year longer in the heterogeneous agent maximal equilibrium.

3.5 Equilibria with Time-Invariant Heterogeniety

The model outlined above has consumer heterogeneity of a particular type, namely only transient, idiosyncratic heterogeneity that is due to the *IID* preference shocks. This idiosyncratic heterogeneity is sufficient to generate trade and equilibrium even in the presence of transactions costs, but it does not enable the model to capture the larger gains from trade from the operation of a secondary market that come from trade between consumers with persistent differences in preferences for cars. These persistent differences can be captured by allowing for different *types* of consumers, which we denote by τ. The types can index permanent differences in consumer preferences, which we denote by utility functions indexed both by the age of the car a and the type of the consumer, τ, given by $u_\tau(a)$. In Appendix 1 we also cover the intermediate case where there is time-varying, idiosyncratic heterogeneity similar to the

Figure 1: Example Stationary Equilibrium



ε_t shocks, but serially correlated over time for a given consumer even though we continue to assume independence in the shocks across different consumers.

In our continuum consumer economy let there be a finite number of types $\{\tau_1,\dots,\tau_n\}$ and let $f(\tau_i)$ be the fraction of consumers of type τ_i, $i=1,\dots,n$. Let $v_\tau(d,a)$ be the decision-specific value function for a type τ consumer, and $\Pi_\tau(d|a)$ be the corresponding choice probability for these consumers. In addition, let $\Delta_\tau(P)$ be the trade transition probability matrix for a consumer of type τ when the price vector is P, given by formula (51) above. In a stationary equilibrium, consumers will end up "specializing" in their holdings of different ages of automobiles, and there may be differences across types of consumers in the fraction of each type that holds the outside good. To account for this and define an equilibrium with a finite number of different types of consumers, let $q_\tau(P)$ be the invariant probability distribution for the transition probability matrix $\Delta_\tau(P)\Omega$, i.e.

$$q_\tau(P) = q_\tau(P)\Delta_\tau(P)\Omega. \tag{66}$$

That is, in a stationary equilibrium, the distribution of holdings of consumers of type τ will be given by the solution $q_\tau(P)$ to equation (66). The aggregate stationary holdings distribution $q(P)$ will be given by

$$q(P) = \sum_\tau q_\tau(P)f(\tau). \tag{67}$$

We will need to show that in a stationary equilibrium we have $q(P) = q(P)\Omega$, i.e. the aggregate distribution of holdings is an invariant distribution with respect to the aging/accident transition probability Ω. But before demonstrating this, we define the equilibrium price vector P as the value of P that sets aggregate excess demand, $ED(P) = 0$, similar to the previous sections. However with different types of consumers we must calculate supplies and demands weighting by the probability $f(\tau)$ of each type of consumer, in recognition that different types consumers have different trading behaviors and also different stationary holdings distributions in equilibrium.

Let $S(a,P)$ be the aggregate supply of cars of age a, $a=1,\dots,\overline{a}-1$ when the price is P. This is given by

$$S(P)(a) = \sum_\tau q_\tau(a,P)[1-\Pi_\tau(-1|a,P)]f(\tau) \tag{68}$$

where $q_\tau(a,P)$ is the fraction of cars of age a held by consumers of type τ implied by the invariant distribution $q_\tau(P)$ given in equation (66). Similarly let $D(a,P)$ be the aggregate demand for cars of age $a = 1, \ldots, \overline{a} - 1$ given by

$$D(P)(a) = \sum_{\tau} \left[\Pi_\tau(a|\emptyset, P) q_\tau(\emptyset, P) + \sum_{a'=1}^{\overline{a}} \Pi_\tau(a|a', P) q_\tau(a', P) \right] f(\tau) \tag{69}$$

Then $S(P)$ and $D(P)$ are the $\overline{a} - 1 \times 1$ vectors whose elements are given by formulas (68) and (69), respectively. Note that equation (66) and the fact that the first column of Ω is the first basis vector $e_1 = (1, 0, \ldots, 0)'$ implies that for each τ we have

$$q_\tau(\emptyset, P) = \Pi_\tau(\emptyset|\emptyset, P) q_\tau(\emptyset, P) + \sum_{a'=1}^{\overline{a}} \Pi_\tau(\emptyset|a', P) q_\tau(a', P), \tag{70}$$

which can be interpreted as "the excess demand for the outside good is 0, for each type τ." Equation (70) can be solved for a unique solution $q_\tau(\emptyset, P)$ similar to equation (57) in the case of only a single time invariant type of consumer in section 3.2. This implies that for each consumer type τ, when $q_\tau(\emptyset, P)$ is given by the solution to (70) then $q_\tau(P)$ will be uniquely determined invariant distribution to $\Delta_\tau(P)\Omega$.

Summing equation (70) over all consumer types we have

$$\sum_{\tau} q_\tau(\emptyset, P) f(\tau) = \sum_{\tau} \left[\Pi_\tau(\emptyset|\emptyset, P) q_\tau(\emptyset, P) + \sum_{a'=1}^{\overline{a}} \Pi_\tau(\emptyset|a', P) q_\tau(a', P) \right] f(\tau). \tag{71}$$

Thus we have shown that, in equilibrium, the following equation holds for $a = 1, 2, \ldots, \overline{a} - 1$ and $a = \emptyset$

$$\sum_{\tau} q_\tau(a, P) f(\tau) = \sum_{\tau} q_\tau(P) \Delta_\tau(P)(a) f(\tau) \tag{72}$$

Since $q_\tau(P)$ and $q_\tau(P)\Delta_\tau(P)$ are probability distributions for each consumer type τ, they must sum to 1. It follows that $\sum_\tau q_\tau(P) f(\tau)$ and $\sum_\tau q_\tau(P)\Delta_\tau(P) f(\tau)$ are also probability distribuitons which also sum to 1. If follows from this and equation (72) that

$$\sum_{\tau} q_\tau(\overline{a}, P) f(\tau) = \sum_{\tau} \left[\Pi_\tau(0|\emptyset, P) q_\tau(\emptyset, P) + \sum_{a'=1}^{\overline{a}} \Pi_\tau(0|a', P) q_\tau(a', P) \right] f(\tau). \tag{73}$$

which states that in equilibrium the market is in *flow equilibrium* — i.e. the demand for new cars equals the supply of cars of age $\overline{a}$ which are scrapped. We summarize this discussion as

Theorem 4 *If a stationary equilibrium exists to the economy with consumers with time invariant* *heterogeneity, then $q(P)$ is an invariant distribution*

$$q(P) = \sum_{\tau} q_{\tau}(P) f(\tau) = \sum_{\tau} q_{\tau}(P)\Delta_{\tau}(P) f(\tau) = q(P)\Omega. \tag{74}$$

Proof: The equilibrium condition that $ED(P) = 0$ implies that the first equation in (74) holds. Using that equation, post-multiply q by Ω to get

$$q(P)\Omega = \sum_{\tau} q_{\tau}(P)\Delta_{\tau}(P) f(\tau)\Omega = \sum_{\tau} q_{\tau}(P)\Delta_{\tau}(P)\Omega f(\tau) = \sum_{\tau} q_{\tau}(P) f(\tau) = q(P). \tag{75}$$

□

Our approach to solving an equilibrium changes in the presence of time-invariant heterogeneity. To continue to be able to use Newton's method to solve the system of equations $ED(P) = 0$ we must verify smoothness of $q_{\tau}(P)$ in P, i.e. that $q_{\tau}(P) = q_{\tau}(P)\Delta_{\tau}(P)\Omega$ is a well-defined implicit function of P that is continuously differentiable in P. This is not immediately evident, since a straightforward application of the Implicit Function Theorem would involve writing $q_{\tau}(P)$ as a zero of the following system of equations

$$q_{\tau}(P)[I - \Delta_{\tau}(P)\Omega] = 0 \tag{76}$$

However since $\Delta_{\tau}(P)\Omega$ is a transition probability matrix, it is easy to see that ie e is an $(\overline{a}+1) \times 1$ vector of 1s that $[I - \Delta_{\tau}(P)\Omega]e = 0$ so the matrix $[I - \Delta_{\tau}(P)\Omega]$ is singular, which violates a key condition needed for the application of the Implicit Function Theorem to gaurantee the existence and continuity of $\nabla_P q_{\tau}(P)$.

Lemma 3 *For each consumer type τ, $q_{\tau}(P) = q_{\tau}(P)\Delta_{\tau}(P)\Omega$ is the unique invariant distribution to the transition probability matrix $\Delta_{\tau}(P)\Omega$ and it is a continuously differentiable function of P.*

Proof: We have already shown that $q_{\tau}(P)$ is the unique invariant distribution to $\Delta_{\tau}(P)\Omega$. The uniqueness of this invariant distributino enables us to write $q_{\tau}(P)$ as the solution to a system of linear equations that is related to, but distinct from the linear system (76) with the key difference being that the matrix in this related system of linear equations is invertible for each P. This means that we can write $q_{\tau}(P)$ explicitly in terms of the inverse of this matrix, which can be shown by inspection to be a function of P since it depends on P only via the matrix $\Delta_{\tau}(P)$ which is continuously differentiable in P by Lemma 2. Hence, using matrix calculus we can write an explicit formula for the gradient $\nabla_P q_{\tau}(P)$ which we use to implement Newton's method to solve the system of equations $ED(P) = 0$. The details of this are contained in Appendix 2. □

Note that it will not generally be the case that $q_\tau(P) = q_\tau(P)\Delta_\tau(P)$, even in stationary equilibrium. This condition is equivalent to the condition that *excess demand for each type* τ *equals zero* and this will not generally hold if there is net trade between different types of consumers in the economy. The only requirement that equilibrium imposes is that aggregate demand (for all types) equals the aggregate supply of each type of car of each age $a = 1,\ldots,\overline{a}-1$. There may be a net excess demand or supply of cars by different types of consumers reflecting patterns of specialization and gains from trade across different types of consumers. For example, poor consumers may choose to specialize in holding older cars while rich consumers specialize in holding newer cars. We will illustrate such an equilibrium below, and in this equilibrium the rich consumers will be net demanders of newer cars and net suppliers of older cars, and the poorer consumers will be net demanders of older cars but not of newer cars. Most of the trade between rich and poor consumers occurs for cars of roughtly middle ages: the rich supply their middle aged cars to the poor consumers and supply and demand will be equal in aggregate, but the supply of middle aged cars of the rich will not equal the demand by the rich for these cars, and similarly the demand for middle aged cars by poor consumers will not equal the supply of these cars by other poor consumers.

We summarize this discussion in Theorem 5.

Theorem 5 *If a stationary equilibrium exists to the economy with consumers with time invariant heterogeneity given above, then in general we have*

$$q(P) \neq q(P)\Delta(P) \quad \text{and} \quad q(P) \neq q(P)\Delta(P)\Omega \tag{77}$$

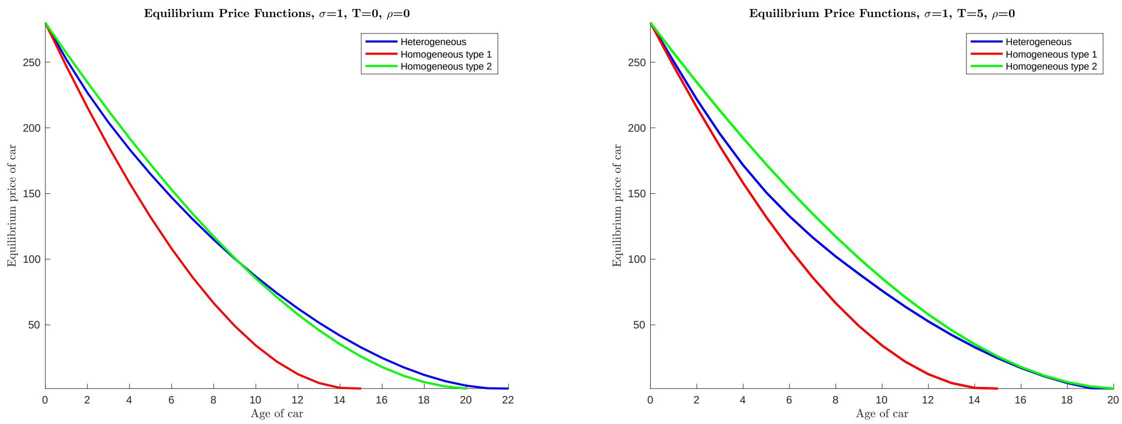
where

$$\Delta(P) = \sum_\tau \Delta_\tau(P) f(\tau). \tag{78}$$

Comparing Theorem 5 (for an economy with mulitple time invariant types of consumers) and Theorem 2 (for an economy with only one time invariant type of consumer) the condition for "demand equilibrium" $q(P) = q(P)\Delta(P)$ holds in the homogenous type case but not in general in the heterogeneous type case. However the condition that $q(P) = q(P)\Omega$, i.e. the aggregate holdings distribution of cars is an invariant distribution of Ω, holds in both cases.

Figure 2 illustrates some of the equilibrium in a heterogeneous agent economy with two types of consumers. We set the discount factor for all agents to $\beta = .95$ and normalized the σ parameter for the extreme value shocks affecting utility (which can be interepreted as a type of *IID* time-varying heterogeneity) to $\sigma = 1$. Consumers of type τ_1 have utility func-

Figure 2: Equilibrium Price Functions in a Two Type Economy

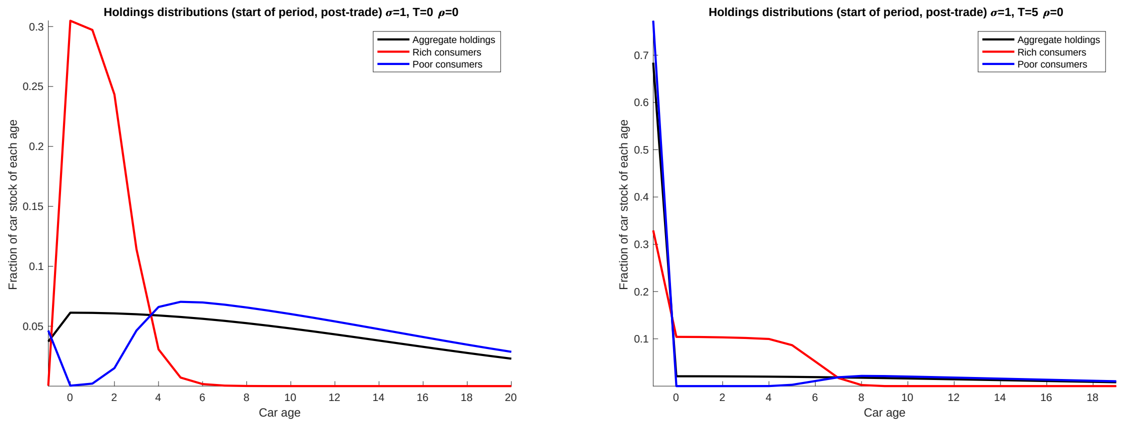


tions $u(a,y,\tau_1) = 45 - 3.2a - y$ where y represents non-car consumption. Their utility for the outside good is 0. Type τ_2 consumers have utility function $u(a,y,\tau_2) = 55.1 - 2.7a + 1.5y$ and have a utility for the outside good which is also normalized to 0. Thus, type τ_1 consumers have a lower "marginal utility of money" (i.e. $\mu = 1$) than type τ_2 consumers, and they also have a stronger preference for cars as reflected by the fact that the utility decreases more quickly with the age of car compared to type τ_2 consumers.

Thus it seems reasonable to conjecture that type τ_1 consumers are akin to "richer" consumers who would be more likely to own newer cars in equilibrium, whereas type τ_2 consumers can be considered as "poorer" consumers who are likely to own older cars in equilibrium. However it is not clear *a priori* whether what relative fractions of type τ_1 and τ_2 consumers will choose to have no car. Notice that type τ_2 consumers actually obtain higher utility from sufficiently new cars, if we ignore the disutility of money (sacrifice of consumption of other goods) needed to buy newer cars. So the question of which types of cars these two different types of consumers will hold is an interesting one and can be answered by calculating an equilibrium and displaying the equilibrium holdings distributions q_τ for the two consumer types.

Figure 2 plots two equilibria, one for the case where there are zero transactions costs, $T = 0$, and the other for the case where $T = 5$. The hetergenous agent equilibrium is plotted as the blue line in both panels of figure 2. For comparison we also plot price functions for zero transaction cost homogenous agent economies, and the red line plots the equilibrium for an economy with only type 1 (rich) consumers adn the green line is the equilibrium in an economy with only type 2 (poor) consumers. The heterogenous agent equilibrium has 20% rich consumers and 80% poor consumers.

Figure 3: Equilibrium Holdings in a Two Type Economy



Notice that when transactions costs are zero, there is an equilibrium with $\overline{a} = 22$ but when $T = 5$ the maximum supportable equilibrium has $\overline{a} = 20$: thus the higher transactions costs have succeeded in partially "killing off" the market for used cars. Also note how the heterogenous agent equilibrium (blue line) starts out closer to the homogeneous agent equilibrium with only rich consumers (red line) for relatively new cars, but approaches the homogenous agent equilibrium prices in an economy with only poor consumers as car ages increase. This is due to the pattern of specialization in car holdings that we illustrated in figure 3 below.

The self-sorting of the two types of consumers into the ages of cars they hold in equilibrium is obvious from figure 3. The "rich" type τ_1 consumers hold the newest cars and in particular are much more likely to buy new cars than the poorer type τ_2 consumers. In addition, the poor are much more likely not to own a car compared to the rich. These findings are ones we might expect, where there is "specialization in holdings" that enables gains from trade between the two types of consumers: the rich consumers buy brand new cars and hold them for several years and then sell them to poor consumers who also hold them for several years, trading the cars over a succession of poor owners until the car is scrapped.

The effect of transactions costs on the holdings and trade in secondary markets for cars is also evident in figure 3: as transactions costs increase, fewer consumers hold cars and more choose the outside good. This reduces the demand for new cars and the aggregate holdings of cars in the economy.

We believe that very rich patterns of holding cars and trading cars can be obtained from this relatively simple dynamic equilibrium model. We believe it can be extended in various interesting directions, such as allowing for multiple types of cars and oligopoly competition in the new car market, where oligopolists consider not only the competition from other car manufacturers, but also the "competition" provided by their own used cars, similar to the lines of Esteban and Shum (2007). The model could be extended further by allowing oligopolists to make product quality and durability choices as well as pricing decisions, and it is potentially possible to allow for the presence of rental intermediaries as well. We will consider several of these extensions in section 5.

3.6 Equilibrium with different types of cars

Suppose there are J different types of cars (e.g. makes/models), with corresponding new car prices $\overline{P}_j$, and scrap prices $\underline{P}_j$, $j = 1, \ldots, J$. We continue to treat new car prices as fixed, consistent with the "small open economy assumption" and continue to assume an infinitely elastic demand for cars for their scrap value, though at perhaps different values depending on the car type. In this section we derive equations characterizing a stationary equilibrium in a market with $J > 1$ different types of cars. For notational reasons we will let the state and decision corresponding to the outside good (i.e. not owning a car) be denoted by $j = \emptyset$ and the decision to keep the current car be denoted by $j = 0$ (as opposed to $a = -1$ in our analysis of equilibrium with only one car type).

When there are multiple car types, it is reasonable adopt a *nested multinomial logit* (NMNL) specification for choice probabilities following Berkovec (1985). The motivation for a NMNL specification compared to the standard multinomial logit (MNL) specification for preference shocks used in the previous sections is to avoid the problem of *independence from irrelevant alternatives* (IIA) that is implied by the joint independence of the random preference shocks implicit in the MNL specification for choice probabilities. This implies an undesirable pattern of equal cross-elasticities in choice probabilities that Berry, Levinsohn and Pakes (1995) have emphasized. In addition, in a choice problems with large numbers of possible alternatives, the MNL specification is undesirable due to the assumption that the random preference shocks have *IID* Type 1 extreme value distributions. We would expect alternatives that are more similar in terms of their observed characteristics (such as two different ages of the same make/model of car) have preference shocks that are more correlated with each other compared to different ages of completely different types of cars. The independence property can therefore lead to excessive switching between different types of cars induced purely by chance realizations of the random preference shocks. In an NMNL specification, we can control the variability and correlation of the random preference shocks to reflect natural patterns of similarity, such as similar unobserved attributes of different ages of the same type (e.g. make/model) of car. As a result the NMNL does not exhibit the IIA property of the MNL model, and the variability in random preference shocks can be more flexibly controlled to avoid excessive switching behavior. This can produce a higher degree of persistence in choices such as keeping the same car longer, and thereby reflecting "brand loyalty" at times when a consumer does decide to trade their old car for a new one.

The NMNL model is implied by a multivariate distribution for the vector ε of idiosyncratic shocks to consumer utility corresponding to the different discrete choices a consumer can make. Consider a consumer who does not own a car, so their car state is denoted by $\emptyset$. The choice set for this consumer is $D(\emptyset) = \{\emptyset, \{(j,d)|d = 0,\ldots,\overline{a}_j - 1, j = 1,\ldots,J\}\}$, where the choice $j = \emptyset$ corresponds to remaining in the no car state, and the choice (j,d) corresponds to buying a car of type j and age d. Similarly, the choice set for an individual who has a car (j,a) of type j and age a is given by $D(j,a) = \{\emptyset, 0, \{(j',d)|d = 0,\ldots,\overline{a}_{j'} - 1, j' = 1.\ldots,J\}\}$ where $j = \emptyset$ corresponds to the decision to purge their current car and own no car next period, $j = 0$ corresponds to the decision to keep their current car and the choice pair (j',d) corresponds to the decision to trade their current car for another car of type j' and age d. Let ε be a vector whose indices are given by either the indices in $D(\emptyset)$ (if the individual does not have a car) or $D(j,a)$ (if the individual has a car of type j and age a).

McFadden (1981) introduced a class of multivariate extreme value distributions that include collections of *IID* Type 1 extreme value distributions as a special case. He called this class the generalized extreme value family (GEV) and showed it could be represented as a joint CDF of the form

$$F(\varepsilon_1,\ldots,\varepsilon_n) = \exp\{-G(e^{-\varepsilon_1},\ldots,e^{-\varepsilon_n})\} \tag{79}$$

where the function G maps the positive orthant of R^n into the positive real line, and satisfies several properties, including homogeneity of degree μ for some $\mu > 0$. Note that the CDF of a collection of *IID* Type 1 extreme value random variables is a special case of equation (79) when $G(y_1,\ldots,y_n) = \sum_{i=1}^n y_i$.

In the auto problem, we index the arguments of the joint CDF as $(\varepsilon_\emptyset, \varepsilon_0, \{\varepsilon_{d,j}\})$ where $\varepsilon_\emptyset$ is a random preference shock associated with the decision to chooce the outside good ($j = \emptyset$), ε_0 is a random preference shock associated with the decision to keep the current car ($j = 0$), and $\{\varepsilon_{d,j}\}$ are the preference shocks associated with different possible choices of car types and ages, (d, j). Thus, $F(\varepsilon_\emptyset, \varepsilon_0, \{\varepsilon_{d,j}\})$ is the CDF of these preference shocks and we assume it has the representation

$$F(\varepsilon_\emptyset, \varepsilon_0, \{\varepsilon_{d,j}\}) = \exp\{-G(e^{-\varepsilon_\emptyset}, e^{-\varepsilon_0}, \{e^{-\varepsilon_{d,j}}\})\} \tag{80}$$

where the function $G(y_\emptyset, y_0, \{y_{d,j}\})$ is given by

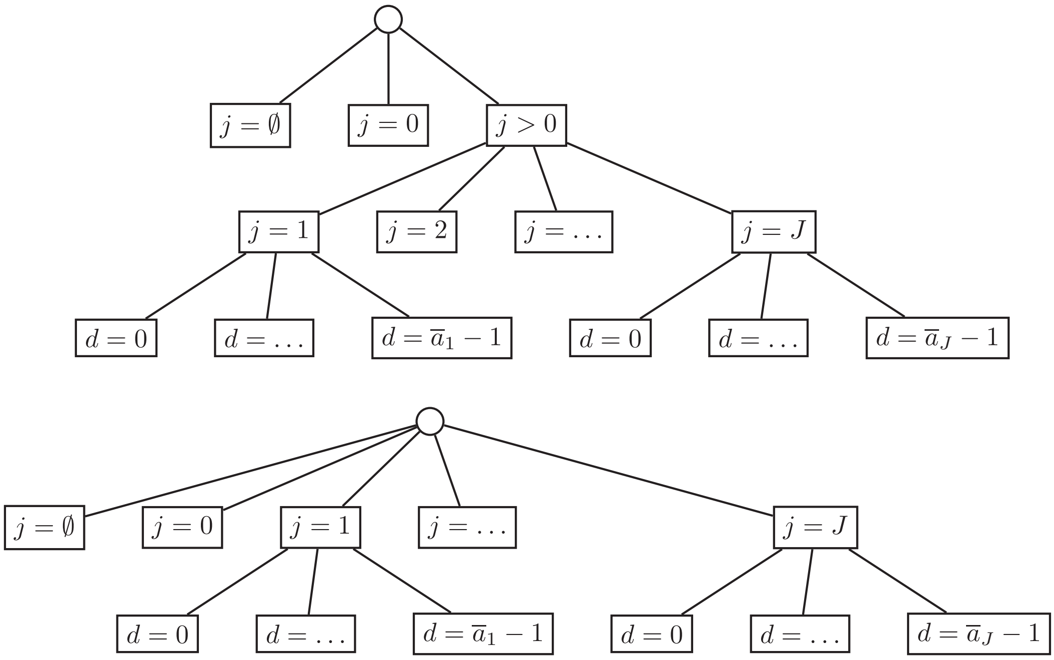
$$G(y_\emptyset, y_0, \{y_{d,j}\}) = y_\emptyset^{\frac{1}{\sigma}} + y_0^{\frac{1}{\sigma}} + \left(\sum_{j=1}^{J} \left(\sum_{d=0}^{\overline{a}_j - 1} y_{d,j}^{\frac{1}{\sigma_j}} \right)^{\frac{\sigma_j}{\sigma_{j>0}}} \right)^{\frac{\sigma_{j>0}}{\sigma}}. \tag{81}$$

where σ, $\sigma_{j>0}$ and $\{\sigma_j\}$, $j = 1,\ldots,J$ are $J+2$ "scale/similarity parameters" of the GEV distribution that control the scale of the random preference shocks as well as the degree of correlation in these shocks for subsets of car choices that we will explain below. To be a valid CDF, the similarity parameters must satisfy the inequalities

$$\sigma \geq \sigma_{j>0} \geq \sigma_j \geq 0 \quad j = 1,\ldots,J \tag{82}$$

The top panel of figure 4 illustrates the patterns of correlation and similarity in the random preference shocks implied by the nested logit GEV specification for F in equations (80) and (81). It take the form of three level choice tree, where at the top level of the choice tree there are only 3 alternatives: a) choose to have no car, $j = \emptyset$, b) choose to keep your current car, $j = 0$, or c) trade your car for another car of either the same or different type $j \in \{1,\ldots,J\}$, denoted by the branch $j > 0$ in figure 4. If a person chooses to trade, $j > 0$ then there is a two level nested logit sub-tree descending below the $j > 0$ branch. There will be a similarity parameter $\sigma_{j>0}$ scaling the shocks at the upper level of this subtree, as well as the car type specific similarity parameters σ_j discussed above. By choosing both $\sigma_{j>0}$ and σ_j to be sufficiently small, we can reduce the effects of random shocks in current utility on car switching behavior, and thus the model can explain obdserved behavior using a lower level of transactions costs when these similarity parameters are sufficiently small. However the fact that there are these two ways to model car switching (i.e. either with high transactions costs in a two level nested logit model or lower transactions costs in a three level nested logit model where the additional parameters $\sigma_{j>0}$ and σ_j, $j \in \{1,\ldots,J\}$ are chosen to be sufficiently small) raises the concern about the identification of these two different explanations for the dynamics of car switching behavior. Note that the parameter σ controls the level of variability of the preference shocks in the top level of the choice tree. As $\sigma \to 0$ the overall scale of the preference shocks tend to zero due

Figure 4: Choice trees for alternative nested logit specifications of the GEV distribution for ε



to the inequalities (82). If we impose the restriction that $\sigma = \sigma_{j>0}$, then the three level NMNL choice tree collapses to the two level choice tree illustrated in the bottom panel of figure 4 where the scale parameters preference shocks for the top level choices are the same, and we only allow different scale parameters σ_j to reflect correlation/similarity in different ages of the them type j of car. If we further restrict that $\sigma = \sigma_{j>0} = \sigma_j$ for $j = 1, \ldots, J$, then the NMNL collapses to a standard MNL model, where there is a common scale parameter σ for all preference shocks, but these shocks are *IID*. Below we derive formulas for the value functions under the general three level NMNL specification illustrated in the top panel of figure 4.

Let $V_\tau(\emptyset, \varepsilon)$ be the discounted utility of a consumer τ who does not own a car. Since τ denotes the type of consumer which is time-invariant, we will suppress it to space in the equations below. The Bellman equation for this state is given by

$$V_\tau(\emptyset,\varepsilon) = \max\left[v_\tau(\emptyset,\emptyset) + \varepsilon(\emptyset), \max_{j\in\{1,\ldots,J\}} \max_{d\in\{0,1,\ldots,\overline{a}_j-1\}} [v_\tau(d,j,\emptyset) + \varepsilon(d,j)]\right], \tag{83}$$

where $\overline{a}_j$ is the scrappage age for car type j and

$$\begin{aligned} v_\tau(d,j,\emptyset) &= u_\tau(d,j) - \mu[P(d,j) + T(P,d,j,\emptyset)] + \beta\left[(1-\alpha_j(d))EV_\tau(d+1,j) + \alpha_j(d)EV_\tau(\overline{a}_j,j)\right] \\ v_\tau(\emptyset,\emptyset) &= u_\tau(\emptyset) + \beta EV_\tau(\emptyset), \end{aligned} \tag{84}$$

where $u_\tau(d,j)$ and $EV_\tau(d,j)$ are the current period utility and expected future utility that a consumer obtains from owning a car of type j and age d, $\alpha_j(d)$ is the probability that a car of type j and age d has an accident that results in a total loss, and $EV_\tau(\emptyset)$ is the conditional expectation of $V(\emptyset,\varepsilon)$, which represents the expectation of future utility for a consumer who does not currently own a car. When ε is a vector of *IID* GEV preference shocks, the expected value has a different formula than the one given in the logit case in equation (43).

$$EV_\tau(\emptyset) = \sigma \log\left(\exp\{v_\tau(\emptyset,\emptyset)/\sigma\} + \exp\{I_{j>0}(\emptyset)/\sigma\}\right) \tag{85}$$

where $I_{j>0}(\emptyset)$ is the *inclusive value* or *ex ante* expected maximized utility corresponding to the choice of moving out of the no car state and buying some car (d,j) that provides the consumer the *ex post* highest discounted utility after observing the preference shocks for each (d,j) alternative, and is given by

$$I_{j>0}(\emptyset) = \sigma_{j>0} \log\left(\sum_{j=1}^{J} \exp\{I_j(\emptyset)/\sigma_{J>0}\}\right). \tag{86}$$

where $I_j(\emptyset)$ is the inclusive value or expected maximal discounted utility associated with the choice of a particular car type j by an individual who does not own a car, and is given by

$$I_j(\emptyset) = \sigma_j \log\left(\sum_{d=0}^{\overline{a}_j-1} \exp\{v_\tau(d,j,\emptyset)/\sigma_j\}\right). \tag{87}$$

Similarly, we extend the Bellman equation for a consumer who owns a car of type j and age a to allow for the option to "purge" their car, i.e. sell the car but not buy another to replace it:

$$\begin{aligned} V_\tau(a,j,\varepsilon) &= \max\Big[v_\tau(\emptyset,a,j)+\varepsilon(\emptyset), v_\tau(0,a,j)+\varepsilon(0), \\ &\qquad \max_{j'\in\{1,\dots,J\}} \max_{d\in\{0,1,\dots,\overline{a}_{j'}-1\}} [v_\tau(d,j',a,j)+\varepsilon(d,j')]\Big], \end{aligned}$$

where $v_\tau(\emptyset,a,j)$ is the value of selling one's current car of type j age a and not replacing it

$$v_\tau(\emptyset,a,j) = u_\tau(\emptyset) + \mu P(a,j) + \beta EV_\tau(\emptyset). \tag{88}$$

and $v_\tau(0,a,j)$ is the value of keeping the current car (a,j) given by

$$v_\tau(0,a,j) = u_\tau(a,j) + \beta\left[(1-\alpha_j(a))EV_\tau(a+1,j) + \alpha_j(a)EV_\tau(\overline{a}_j,j)\right], \tag{89}$$

and $v_\tau(d,j',a,j)$ is the value of trading one's current car (a,j) for another car (d,j').

$$\begin{aligned} v_\tau(d,j',a,j) &= u_\tau(d,j') - \mu[P(d,j') - P(a,j) + T(P,d,j',a,j)] + \\ & \beta\left[(1-\alpha_{j'}(d))EV_\tau(d+1,j') + \alpha_{j'}(d)EV_\tau(\overline{a}_{j'},j')\right]. \end{aligned} \tag{90}$$

The expected value of state (a,j) is $EV_\tau(a,j)$ given by

$$EV_\tau(a,j) = \begin{cases} \sigma\log\left(\exp\{v_\tau(\emptyset,a,j)/\sigma\} + \exp\{I_{j>0}(a,j)/\sigma\}\right) & \text{if } a = \overline{a}_j \\ \sigma\log\left(\exp\{v_\tau(\emptyset,a,j)/\sigma\} + \exp\{v_\tau(0,a,j)/\sigma\} + \exp\{I_{j>0}(a,j)/\sigma\}\right) & \text{otherwise} \end{cases} \tag{91}$$

where the first equation for $EV_\tau(a,j)$ in (91) is for the case where the current car is at the scrappage age threshold, $a = \overline{a}_j$, so keeping this car is no longer an option, and the second formula is for the case where $a < \overline{a}_j$ so the consumer has the additional option to choose to keep the current car (a,j) rather than trade it, which has the value $v_\tau(0,a,j)$. The inclusive value for the decision to trade for another car is $I_{j>0}(a,j)$ given by

$$I_{j>0}(a,j) = \sigma_{j>0}\log\left(\sum_{j'=1}^{J}\exp\{I_{j'}(a,j)/\sigma_{j>0}\}\right) \tag{92}$$

where $I_{j'}(a,j)$ is the inclusive value associated with the decision to trade the current car (a,j) for some car (d',j') of car type j' given by

$$I_{j'}(a,j) = \sigma_{j'}\log\left(\sum_{d'=0}^{\overline{a}_{j'}-1}\exp\{v_\tau(d',j',a,j)/\sigma_{j'}\}\right). \tag{93}$$

For notational simplicity, we suppressed the dependence of the value functions on the set of car prices. However to describe equilibrium it is now necessary to make the dependence on prices P explicit. Let P be the concatenated vector of prices for all ages of traded used cars, $a_j \in \{1,2,\ldots,\overline{a}_j - 1\}$, $j \in \{1,2,\ldots,J\}$

$$P = (P_1', P_2', \ldots, P_J')' \tag{94}$$

where P_j is the $\overline{a}_j - 1 \times 1$ vector of age specific prices of car type j, i.e. $P_j(a)$ is the price of a car of type j of age a. As in the previous sections, all of the key objects of the model, the expected values EV_τ and value functions v_τ and the implied choice probabilities Π_τ will be implicit functions of P. Define $\overline{a}$ as

$$\overline{a} \equiv \sum_{j=1}^{J} \overline{a}_j \tag{95}$$

Thus, P is the $\overline{a} - J \times 1$ vector of prices of all traded used cars in the economy that must be determined in a stationary equilibrium of the model.

Similar to our discussion of the case of only one car type, the system of equations (85) and (91) are the equivalent of the Bellman equation, but after the *GEV* distributed preference shocks ε have been integrated out. As we previously discussed the system defines EV_τ as the unique fixed point to a contraction mapping, and EV_τ can also be calculated using Newton iterations exactly as described in equation (47) in the case of a single car type. We can also use the Implicit Function Theorem to show that EV_τ is a smooth implicit function of P which we denote as $EV_\tau(P)$. From the solution EV_τ we can construct the choice-specific values $v(d, j', a, j)$ which are also implicit functions of P and where appropriate we write $v(d, j', a, j, P)$ to emphasize this.

However unlike the choice probability formulas in the previous sections, the GEV CDF for $F(\varepsilon)$ implies the following *nested logit* choice probabilities for a consumer who does not own a car, $j = \emptyset$. First consider the probability that the person chooses to remain in the no car state. This has probability $\Pi(\emptyset|\emptyset, P)$ given by

$$\Pi_\tau(\emptyset|\emptyset, P) = \frac{\exp\{v_\tau(\emptyset, \emptyset)/\sigma\}}{\exp\{v_\tau(\emptyset, \emptyset)/\sigma\} + \exp\{I_{j>0}(\emptyset)/\sigma\}} \tag{96}$$

Let $\Pi_\tau(d, j|\emptyset, P)$ be the probability that the consumer chooses to buy a car (d, j). This is given by the product of "transition probabilities" reflecting the three level choice tree

$$\Pi_\tau(d, j|\emptyset, P) = \Pi_\tau(d|j, \emptyset, P)\Pi_\tau(j|j > 0, \emptyset, P)\Pi_\tau(j > 0|\emptyset, P) \tag{97}$$

where $\Pi_\tau(j > 0|\emptyset, P)$ denotes the probability that the consumer chooses some type of car,

$$\Pi_\tau(j > 0|\emptyset, P) = 1 - \Pi_\tau(\emptyset|\emptyset, P) = \frac{\exp\{I_{j>0}(\emptyset)/\sigma\}}{\exp\{v_\tau(\emptyset, \emptyset)/\sigma\} + \exp\{I_{j>0}(\emptyset)/\sigma\}}, \tag{98}$$

and $\Pi_\tau(j|j > 0, \emptyset, P)$ is the probability the consumer chooses car type j given the decision to choose some type of car,

$$\Pi_\tau(j|j > 0, \emptyset, P) = \frac{\exp\{I_j(\emptyset)/\sigma_{j>0}\}}{\sum_{j'=1}^{J} \exp\{I_{j'}(\emptyset)/\sigma_{j>0}\}} \tag{99}$$

and $\Pi_\tau(d|j,\emptyset,P)$ is the probability the consumer chooses a car of age d given that they chose a car of type j

$$\Pi_\tau(d|j,\emptyset,P)=\frac{\exp\{v_\tau(d,j,\emptyset,P)/\sigma_j\}}{\sum_{d'=0}^{\overline{a}_j-1}\exp\{v_\tau(d',j,\emptyset,P)/\sigma_j\}}. \tag{100}$$

Now consider a consumer who has a car (a,j) of age a and type j. The probability of the choice $d=\emptyset$ (i.e. to sell their current car and not replace it with another one) is given by

$$\Pi_\tau(\emptyset|a,j,P)=\begin{cases}\frac{\exp\{v_\tau(\emptyset,a,j,P)/\sigma\}}{\exp\{v_\tau(\emptyset,a,j,P)/\sigma\}+\exp\{v_\tau(0,a,j,P)/\sigma\}+\exp\{I_{j>0}(a,j)/\sigma\}} & \text{if } a<\overline{a}_j\\ \frac{\exp\{v_\tau(\emptyset,a,j,P)/\sigma\}}{\exp\{v_\tau(\emptyset,a,j,P)/\sigma\}+\exp\{I_{j>0}(a,j)/\sigma\}} & \text{if } a=\overline{a}_j\end{cases} \tag{101}$$

which reflects the constraint that for any car type j a consumer who holds the oldest possible age $a=\overline{a}_j$ is forced to scrap it, so we have

$$\Pi_\tau(0|a,j,P)=\begin{cases}\frac{\exp\{v_\tau(0,a,j,P)/\sigma\}}{\exp\{v_\tau(\emptyset,a,j,P)/\sigma\}+\exp\{v_\tau(0,a,j,P)/\sigma\}+\exp\{I_{j>0}(a,j)/\sigma\}} & \text{if } a<\overline{a}_j\\ 0 & \text{if } a=\overline{a}_j\end{cases} \tag{102}$$

Since there are only three possible choices at the top level of the choice tree in figure 4, it follows that the probability of choosing to trade the current car (a,j) for some other car (d,j') is given by

$$\Pi_\tau(j>0|a,j,P)=1-\Pi_\tau(\emptyset|a,j,P)-\Pi_\tau(0|a,j,P) \tag{103}$$

which is valid for any value of $a\in\{1,\dots,\overline{a}_j\}$. Let $\Pi_\tau(d,j'|a,j,P)$ be the probability the consumer trades their current car (a,j) for some other car (d,j'). Similar to the probability $\Pi_\tau(d,j'|\emptyset,P)$ given in equation (97) above, this probability can be written as a product of three "transition probabilities"

$$\Pi_\tau(d,j'|a,j,P)=\Pi_\tau(d|j',a,j,P)\Pi_\tau(j'|j>0,a,j,P)\Pi_\tau(j>0|a,j,P) \tag{104}$$

where the probability of choosing to trade the current car (a,j) for some other age and type of car is $\Pi_\tau(j>0|a,j,P)$ given in equation (103), and $\Pi_\tau(j'|j>0,a,j,P)$ is the probability the consumer chooses car type j' given the decision to trade,

$$\Pi_\tau(j'|j>0,a,j,P)=\frac{\exp\{I_{j'}(a,j,P)/\sigma_{j>0}\}}{\sum_{j'=1}^{J}\exp\{I_{j'}(a,j,P)/\sigma_{j>0}\}}, \tag{105}$$

and $\Pi_\tau(d|j',a,j,P)$ is the probability that the consumer chooses a car of age d given the choice of type of car j'

$$\Pi_\tau(d|j',a,j,P) = \frac{\exp\{v_\tau(d,j',a,j,P)/\sigma_{j'}\}}{\sum_{d'=0}^{\overline{a}_{j'}-1}\exp\{v_\tau(d',j',a,j,P)/\sigma_{j'}\}}. \tag{106}$$

It is important to note that due to the separability implied by our quasi-linear specification of utility combined with a restriction that the transactions cost function is of the form $T(P,d,j')$ depends only on the car the consumer buys, (d,j') but is independent of the type/age (a,j) of car the consumer trades in, implies that the choice of the optimal age of a replacement car will be independent of (a,j) as well, i.e. we have

$$\Pi_\tau(d|j',a,j,P) = \Pi_\tau(d|j',P), \tag{107}$$

so the choice probability for the age of a replacement car satisfies an additional *conditional independence property.* This implies that the only way we can model persistence in replacement choices is mainly on a prospective basis, by insuring that one of the types and ages of possible cars provides significantly higher expected discounted utility than the others. Transactions costs can explain persistence in *current holdings* (i.e. the decision to keep the current car), but conditional on the decision to trade the current car, (i.e. not to keep and not to choose the outside good), the age and type of the currently held car (a,j) has no effect on the consumer's choice of replacement vehicle. The only way we can model such dependence is by allowing the transactions cost function to depend on the current car (a,j) in a non-trivial way, so this would require a more complex specification of the form $T(P,a,j,d,j')$ that depends on (a,j) and not just on (d,j').

In our current specification we have $T(P,a,j,d,j') = T + \rho P(d,j')$ where transactions costs equal the sum of a fixed component T plus a proportional component that is a fraction ρ of the price of the car the consumer buys, $P(d,j')$. However even if we were to change the transactions cost to depend on the difference between the price of the car the consumer purchases and the trade-in value of the car sold, so $T(P,a,j,d,j') = T + \rho(P(d,j') - P(a,j))$, the conditional independence assumption (107) still holds as you can see from examining the formula for the value function for trading, $v_\tau(d,j',a,j)$ in equation (91) which is an additively separable function of its arguments, i.e. we can write it was

$$v_\tau(d,j',a,j) = v_{1\tau}(d,j') + v_{2\tau}(a,j) \tag{108}$$

even in the case where transactions cost are a fraction ρ of the net trade cost, $P(d,j') - P(a,j)$.

Since the $v_{2\tau}(a,j)$ is just an additive term that shifts the discounted utilities of all choices (d,j') the choice probability must be independent of (a,j) consistent with the conditional independence restriction (107).

To complete our analysis, we now consider the definition of stationary equilibrium in a model with multiple car types and types of consumer. We start by writing the equation for excess demand, and we note that with multiple types of cars, each car may have different accident probabilities, so we let $\alpha_j(a)$ denote the accident probability for a car of type j and age a (which leads to the scrappage of the car). Then let $q_j(a)$ be the *conditional invariant distribution* of ages of cars of type j, following the same formula (29) as we derived for the single type car case above.

Let Ω_j be the $\overline{a}_j \times \overline{a}_j$ transition probability matrix representing the aging of a car of type j allowing for accidents, i.e. the transition probability matrix given by formula (33) but with the age-specific accident probabilities $\alpha_j(a)$ given by those specific to car type j. Now consider the $(\overline{a}+1)\times(\overline{a}+1)$ matrix Ω given by

$$\Omega = \begin{bmatrix} \Omega_1 & 0_{1,2} & \cdots & \cdots & 0_{1,J-2} & 0_{1,J-1} & 0_{1,J} & 0_1 \\ 0_{2,1} & \Omega_2 & \cdots & \cdots & 0_{2,J-2} & 0_{2,J-1} & 0_{2,J} & 0_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{J-2,1} & 0_{J-2,2} & \cdots & \cdots & \Omega_{J-2} & 0_{J-2,J-1} & 0_{J-2,J} & 0_{J-2} \\ 0_{J-1,1} & 0_{J-1,2} & \cdots & \cdots & 0_{J-1,J-2} & \Omega_{J-1} & 0_{J-1,J} & 0_{J-1} \\ 0_{J,1} & 0_{J,2} & \cdots & \cdots & 0_{J,J-2} & 0_{J,J-1} & \Omega_J & 0_J \\ 0'_1 & 0'_2 & \cdots & \cdots & 0'_{J-2} & 0'_{J-1} & 0'_J & 1 \end{bmatrix}, \tag{109}$$

where $0_{i,j}$ is a $\overline{a}_i \times \overline{a}_j$ matrix of zeros and 0_j is an $\overline{a}_j \times 1$ vector of zeros. Similar to the case where there is only one type of car, there will be a continuum of stationary distributions to Ω, i.e. a continuum of solutions satisfying $q\Omega = q$. Each solution will take the form

$$q = \sum_{j=1}^{J} \eta_j q_j \tag{110}$$

where $\eta_j \in [0,1]$ and

$$\sum_{j=1}^{J} \eta_j \leq 1 \tag{111}$$

and q_j is the unique invariant distribution to Ω_j, i.e. the unique solution to $q_j = q_j\Omega_j$, $j = 1,\ldots,J$. The parameter η_j represents the "market share" for car type j, i.e. the share of the total population that owns a car of type j, where the residual fraction of the consumers in the economy who choose the outside good is $\eta_\emptyset$ given by

$$\eta_\emptyset = 1 - \sum_{j=1}^{J} \eta_j \tag{112}$$

represents the share of the population who chooses the outside good. Similar to the one car type case, we need additional restrictions to pin down the shares η_j, $j = 1,\ldots,J$. These restrictions come from the equilibrium conditions that specify that the excess demand for each type of car is zero. These additional equilibrium restrictions will then to pick out a solution, i.e. an invariant distribution $q\Omega = q$ from the continuum of possible invariant distributions that constitutes holdings of all ages of the J different cars and the fraction of the population choosing the outside good, $d = \emptyset$.

To do this, we follow the approach of the analysis of a single car type, but generalize it. Define the $(\overline{a}+1)\times(\overline{a}+1)$ transition probability matrix $\Delta_\tau(P)$ by

$$\Delta_\tau(P) = \begin{bmatrix} \Delta_\tau(1|1,P) & \cdots & \Delta_\tau(J-1|1,P) & \Delta_\tau(J|1,P) & \Delta_\tau(\emptyset|1,P) \\ \Delta_\tau(1|2,P) & \cdots & \Delta_\tau(J-1|2,P) & \Delta_\tau(J|2,P) & \Delta_\tau(\emptyset|2,P) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Delta_\tau(1|J-1,P) & \cdots & \Delta_\tau(J-1|J-1,P) & \Delta_\tau(J|J-1,P) & \Delta_\tau(\emptyset|J-1,P) \\ \Delta_\tau(1|J,P) & \cdots & \Delta_\tau(J-1|J,P) & \Delta_\tau(J|J,P) & \Delta_\tau(\emptyset|J,P) \\ \Delta_\tau(1|\emptyset,P) & \cdots & \Delta_\tau(J-1|\emptyset,P) & \Delta_\tau(J|\emptyset,P) & \Delta_\tau(\emptyset|\emptyset,P) \end{bmatrix} \tag{113}$$

where $\Delta_\tau(\emptyset|\emptyset,P)$ is the probability that a consumer who does not own a car at time t will continue not to own a car at time $t+1$ and is given by

$$\Delta_\tau(\emptyset|\emptyset,P) = \Pi_\tau(\emptyset|\emptyset,P) \tag{114}$$

where $\Pi_\tau(\emptyset|\emptyset,P)$ is the probability that a consumer of type τ who has no car chooses to remain in the state of having no car, similar to formula (48) but generalized in the obvious way to allow multiple types of cars.

The generic element of the partitioned matrix $\Delta(P,\tau)$ in equation (113) is $\Delta_\tau(k|j,P)$ which is a matrix of dimension $\overline{a}_j \times \overline{a}_k$ that represents a "sub-transition matrix" of choice probabilites for a consumer who holds an existing car of type j but chooses to buy a car of type k. Let $a_j \in \{1,2,\ldots,\overline{a}_j\}$ be the age of the current car of type j that a consumer of type τ owns and let $a_k \in \{0,1,\ldots,\overline{a}_k - 1\}$ be the age of a car of type k that this consumer chooses to buy instead.

Then we have

$$\Delta_\tau(k|j,P)(a_j,a_k) = \Pi_\tau(a_k,k|a_j,j,P). \tag{115}$$

Note that for technical reasons, we order the columns of $\Delta_\tau(k|j,P)$ with respect to the car age index a_k as we did for the formula for $\Delta_\tau(P)$ in the 1 car type case in equation (51), namely to put the choice of a new car, $a_k = 0$ in the next last column of this matrix and the choice of the outside good in the last column. Then it is possible to show that the analog of Lemma 1 holds in the multi-car case, and the multivariate version of equation (53) holds

$$\nabla_{EV}\Gamma(EV,\tau) = \beta\Delta_\tau(P)\Omega. \tag{116}$$

Now consider the vector $\Delta_\tau(\emptyset|j,P)$. This is a vector of dimension $(\overline{a}_j \times 1)$ whose component a_j is given by

$$\Delta_\tau(\emptyset|j,P)(a_j) = \Pi_\tau(\emptyset|a_j,j,P). \tag{117}$$

Finally, consider the diagonal elements of the partitioned choice probability transition matrix $\Delta_\tau(P)$, $\Delta_\tau(j|j,P)$, $j \in \{1,2,\ldots,J\}$. These are the same as the off-diagonal blocks $\Delta_\tau(i|j,P)$ for $i \neq j$ except that the diagonal blocks are square $(\overline{a}_j \times \overline{a}_j)$ matrices and the diagonal elements of these matrices include the probability of keeping the current vehicle which we recall is denoted by the choice $j' = j = 0$. So we have

$$\Delta_\tau(j|j,P)(a_j,a_j) = \Pi_\tau(a_j,j|a_j,j,P) + \Pi_\tau(0|a_j,j,P), \tag{118}$$

which describes the probability for the two ways of keeping the same age and type of car (a_j,j): 1) trading the current car (a_j,j) for another car (a_j,j), and 2) keeping the current car (a_j,j). The matrix $\Delta_\tau(P)\Omega$ can be written as

$$\begin{bmatrix}
\Delta_\tau(1|1,P)\Omega_1 & \cdots & \Delta_\tau(J-1|1,P)\Omega_{J-1} & \Delta_\tau(J|1,P)\Omega_J & \Delta_\tau(\emptyset|1,P) \\
\Delta_\tau(1|2,P)\Omega_1 & \cdots & \Delta_\tau(J-1|2,P)\Omega_{J-1} & \Delta_\tau(J|2,P)\Omega_J & \Delta_\tau(\emptyset|2,P) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\Delta_\tau(1|J-1,P)\Omega_1 & \cdots & \Delta_\tau(J-1|J-1,P)\Omega_{J-1} & \Delta_\tau(J|J-1,P)\Omega_J & \Delta_\tau(\emptyset|J-1,P) \\
\Delta_\tau(1|J,P)\Omega_1 & \cdots & \Delta_\tau(J-1|J,P)\Omega_{J-1} & \Delta_\tau(J|J,P)\Omega_J & \Delta_\tau(\emptyset|J,P) \\
\Delta_\tau(1|\emptyset,P)\Omega_1 & \cdots & \Delta_\tau(J-1|\emptyset,P)\Omega_{J-1} & \Delta_\tau(J|\emptyset,P)\Omega_J & \Delta_\tau(\emptyset|\emptyset,P)
\end{bmatrix} \tag{119}$$

Define $q_\tau(P)$ as the invariant distribution of the $(\overline{a}+1)\times(\overline{a}+1)$ transition matrix $\Delta_\tau(P)\Omega$, i.e.

$$q_\tau(P) = q_\tau(P)\Delta_\tau(P)\Omega. \tag{120}$$

Thus, $q_\tau(P)$ is a $1\times(\overline{a}+1)$ row vector, representing the stationary holdings distribution for consumers of type τ when the price vector is P (here P is the augmented $(\overline{a}+J)\times 1$ vector that includes both the prices of used cars that are determined endogenously in equilibrium but also the exogenously specified new car prices $\overline{P}_j$ and scrap prices $\underline{P}_j$ for each of the $j\in\{1,\dots,J\}$ types of cars in the market. Note how $q_\tau(P)$ is partitioned/indexed. $q_\tau(\emptyset,P)$ denotes the last element of the vector, which is the fraction of type τ consumers who do not have a car. The other elements are doubly indexed to indicate the conformable partitioning of $q_\tau(P)$ into types and ages of different cars. Thus $q_\tau(a,j,P)$ is the fraction of type τ consumers who hold a car of type j and age a.

We are now ready to define a stationary equilibrium in the case of multiple car types. We now revert to treating the vector P as the $(\overline{a}-J)\times 1$ vector of prices of *used cars* but excluding new cars $a_j=0$ and cars whose age equals the scrappage age $a_j=\overline{a}_j$, since the latter prices are exogenously specified and not determined as part of the equilibrium. A stationary equilibrium price vector is simply a solution P to the system of equations

$$ED(P) = 0 \tag{121}$$

where $ED(P) = (ED_1(P),\dots,ED_J(P))$ where

$$ED_j(P) = D_j(P) - S_j(P) \tag{122}$$

where $D_j(P)$ is the $\overline{a}_j - 1\times 1$ vector given by

$$D_j(P) = \begin{bmatrix} \sum_\tau \left[\Pi_\tau(1,j|\emptyset,P)q_\tau(\emptyset,P) + \sum_{l=1}^{J}\sum_{a=1}^{\overline{a}_l}\Pi_\tau(1,j|a,l,P)q_\tau(a,l,P)\right] f(\tau) \\ \sum_\tau \left[\Pi_\tau(2,j|\emptyset,P)q_\tau(\emptyset,P) + \sum_{l=1}^{J}\sum_{a=1}^{\overline{a}_l}\Pi_\tau(2,j|a,l,P)q_\tau(a,l,P)\right] f(\tau) \\ \cdots \\ \sum_\tau \left[\Pi_\tau(\overline{a}_j-2,j|\emptyset,P)q_\tau(\emptyset,P) + \sum_{l=1}^{J}\sum_{a=1}^{\overline{a}_l}\Pi_\tau(\overline{a}_j-2,j|a,l,P)q_\tau(a,l,P)\right] f(\tau) \\ \sum_\tau \left[\Pi_\tau(\overline{a}_j-1,j|\emptyset,P)q_\tau(\emptyset,P) + \sum_{l=1}^{J}\sum_{a=1}^{\overline{a}_l}\Pi_\tau(\overline{a}_j-1,j|a,l,P)q_\tau(a,l,P)\right] f(\tau) \end{bmatrix}, \tag{123}$$

and where $S_j(P)$ is given by

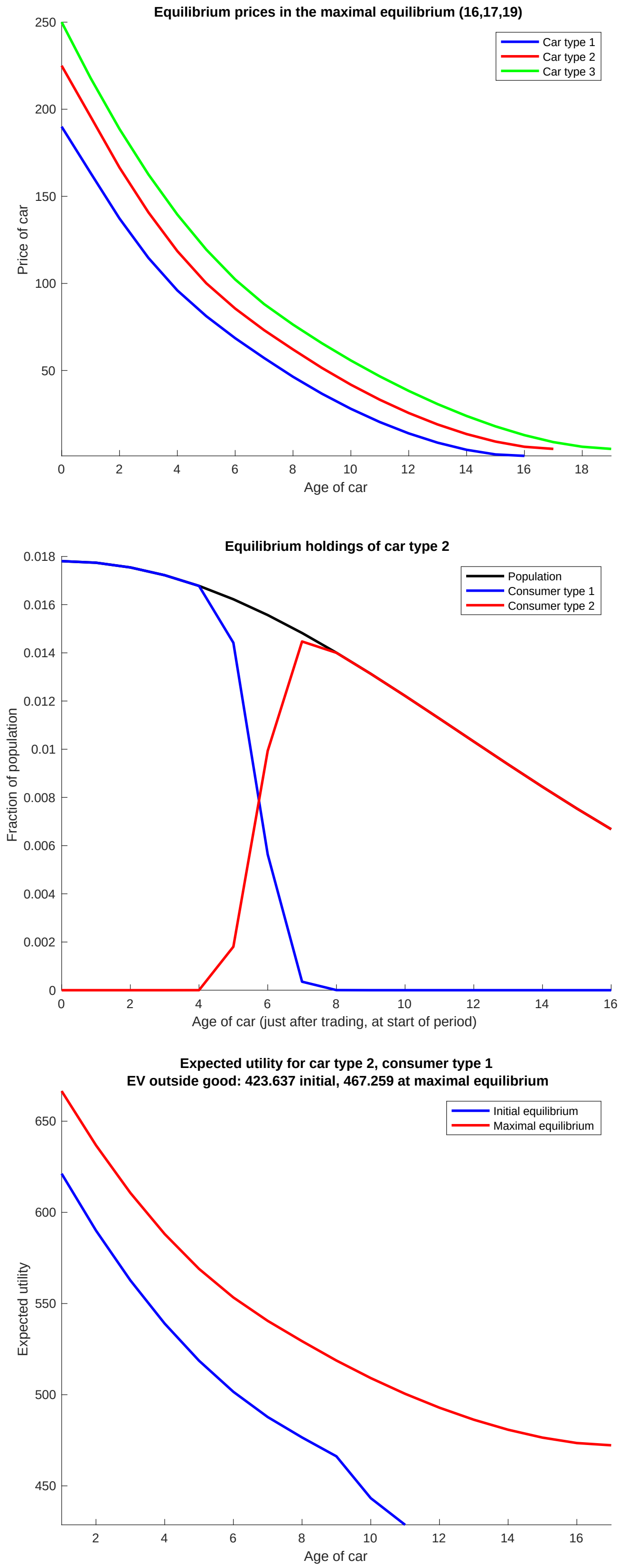
$$S_j(P) = \begin{bmatrix} \sum_\tau [1-\Pi_\tau(-1|1,j,P)]q_\tau(1,j,P)f(\tau) \\ \sum_\tau [1-\Pi_\tau(-1|2,j,P)]q_\tau(2,j,P)f(\tau) \\ \cdots \\ \sum_\tau [1-\Pi_\tau(-1|\overline{a}_j-1,j,P)]q_\tau(\overline{a}_j-1,P)f(\tau) \end{bmatrix}. \tag{124}$$

where $q_\tau(P)$ is the unique invariant distribition to $\Delta_\tau(P)\Omega$ as per (120). We note that Theorem 4 continues to hold in the multiple car type, multiple persistent consumer type case. That is, for the value of P that satisfies $ED(P) = 0$, we we let $q = \sum_\tau q_\tau(P)f(\tau)$ where $q_\tau = q_\tau(P)\Delta_\tau(P)\Omega$ for each consumer type τ. By Theorem 4, $q = q\Omega$ but by Theorem 5, $q \neq q\Delta(P)\Omega$ where $\Delta(P) = \sum_\tau \Delta_\tau(P)f(\tau)$.

Similar to the single car case, we define a maximal equilibrium as the largest vector of scappage ages for the J cars, $(\overline{a}_1,\dots,\overline{a}_J)$, with the property that $P(a,j) \geq \underline{P}_j$ for $a \in \{1,\dots,\overline{a}_j - 1\}$, $j \in \{1,\dots,J\}$. We use the same iterative algorithm that we developed to find a maximal equilibrium in the single car case with the following modification. We solve the social planning problem for a homogeneous consumer economy for each consumer type τ and car type j and take the largest of these socially optimal scrapping thresholds over each consumer type τ and the corresponding shadow prices as our starting guess for prices in the multicar economy. If this initial guess results in a valid equilibrium, we increase the scrappage ages of all car types by 1 simultaneously and compute an updated solution to $ED(P) = 0$ until we find a violation of the price constraint $P(a,j) \geq \underline{P}_j$ for one or more car types j. Then we revert to the largest previous valid value of the vector of scrappage ages $(\overline{a}_1,\dots,\overline{a}_J)$, i.e. the current estimate of the maximal equilibrium. Then we enter a final phase where we increment the scrappage ages of each car type j by 1 individually, keeping the other scrappage ages fixed. We keep doing this until we obtain a violation of the price constraints, and revert to the last estimate of the maximal equilibrium otherwise. When we are no longer able to individually increment the scrappage ages of any of the J car types without violating the price constraint the algorithm terminates, and returns the last estimate of the maximal equilibrium as the stationary equilibrium for the auto market.

Figure 5 illustrates an example maximal equilibrium for an auto market with $J = 3$ car types and 2 consumer types. As a test of the equilibrium, we made the utilities for two of the cars (car types 2 and 3) to be identical for both types of consumers: $u_\tau(a,2) = u_\tau(a,3)$ for all ages a and both consumer types τ. The two types of consumers differ in their marginal utilities of money:

Figure 5: Example Stationary Equilibrium: 3 Car Types and 2 Consumer types



20% of the consumers are "rich" with $\mu_\tau = 1$ and the other 80% are "poor" with $\mu_\tau = 2$. The two consumer types also differ in their preference for the expensive and less expensive cars: the rich consumers obtain higher utility for the more expensive cars and their disutility from owning a car falls faster with age than for the poor consumers. Car types 2 and 3 are the expensive cars but car type 3 is more expensive: $\overline{P}_3 = 250 > \overline{P}_2 = 225$. Thus, we would expect few consumers would choose car type 3 in equilibrium since car type 2 provides the same utility but a new type 2 car is 10% cheaper than a new type 3 car.

Indeed, in the maximal equilibrium computed by our algorithm only 1 tenth of 1% of the population owns type 3 cars. The individuals who do own these cars are induced to buy them due to large idiosyncratic ε shocks. Car ownership is almost equally divided between car types 1 and 2: 22.6% of the population owns the cheaper car type 1 and 22.7% owns the more expensive car type 2. A majority of the population, 54.6%, owns no car. Most of these are the poor consumers: essentially all rich consumers own cars, whereas 68.2% of poor consumers do not own a car.

We also see an endogenous division of car holdings in the middle panel of figure 5: the rich consumers mostly buy new cars of both types and hold them until they are about 5 years old, and then sell them to a sequence of poor consumers who hold the car until it is scrapped. A typical car in this economy will have 3 or 4 different owners. The final panel of the figure shows the gain in discounted utility from the initial guess of the scrappage ages, $(\overline{a}_1, \overline{a}_2, \overline{a}_3) = (10, 11, 12)$ to the final maximal equilibrium at $(\overline{a}_1, \overline{a}_2, \overline{a}_3) = (16, 17, 19)$. Again, we find a strict Pareto gain for all consumers and car types in the move from the initial equilibrium to the maximal equilibrium.

4 Analysis of Danish Car Tax Policy

In this section we illustrate the potential value of the model developed in the previous sections for use in policy analysis. We conduct a simplified, highly stylized analysis of various tax reforms that have been under discussion in Denmark, which has one of the highest tax rates on new cars in the world: 180% prior to 2016 and 150% after 2016 on new cars whose price exceeds approximately $12000. As we noted in the introduction, the Danish government receives between 7 and 11% of all of its tax revenues from this registration tax, and so it is reluctant to simply reduce it without increasing some other tax to make up the difference. In this section we will consider a policy of reducing the new car tax and increasing the fuel tax by a sufficient amount to make the change in taxation approximately budget neutral, using a very simple stylized model of the automobile market in Denmark. In ongoing work we are estimating a large scale econometric model of auto trading and holdings by Danih households in an attempt to provide more refined and accurate forecasts of the welfare and environomental impacts of these types of tax reforms.

In order to do this, we need to extend our model to allow for driving, and to predict how consumers would react to a substantial increase in the gas tax. Obviously, a main reason to have a car is to drive it and the development of our model so far has ignored this key aspect of a car. Let p_j denote the price per kilometer travelled for a car of type j. This equals the price of fuel (e.g. kroners per liter) divided by the car's fuel efficiency (kilometers per liter of gas) for car type j. We assume that fuel efficiency does not degrade with the age of the car and initially we treat the fuel price as fixed (i.e. it does not vary over time). For further simplicity, we consider the case of only one type of car, $J = 1$, below.

Let x denote the number of kilometers a consumer chooses to travel in each period, and let $u_\tau(a,x)$ be the utility a consumer of type τ obtains from owning a car of age a and driving it x kilometers during the period. We make a key assumption

Assumption X *The probability of an accident and other physical deterioration in an automobile is independent of driving, x.*

Though this is a strong and unrealistic assumption, the benefit is that it implies that *driving is a static sub-problem of the consumer's overall dynamic optimization problem.* If we were to relax Assumption X, then the continuous driving choice has dynamic consequences, and the consumer must consider the effect of current driving on the future physical status of their auto. For example, in the automobile model of Rust (1985c) the state of the car is captured by its odometer reading rather than its age, so driving has a direct effect on the depreciation of the car. In our model where the age of a car is the state variable, driving can affect the future value of a car if the probability of an accident is positively related to the amount an individual drives.

Though there is undoubtedly a relationship between the level of driving and the probability of an accident or the future value of the car (as reflected by a higher odometer reading, for any given age of car), we believe that the relationship between current driving decisions and the loss in future value of the car (via depreciation) or via accidents which are undoubtedly related to kilometers driven per period, is sufficiently weak that we believe it does not greatly distort our model to assume that consumers ignore this effect when making their driving decisions.

This implies that driving solves the following *static sub-problem*

$$x(a, p_f, \tau) = \underset{x}{argmax}\, u(a, x, \tau) - \mu(\tau) x p_f \tag{125}$$

We can view $x(a, p_f, \tau)$ as the "demand for driving" function, and besides the price per kilometer p_f it may depend on the age of the car a, especially if it is less pleasant or less safe to drive an older car compared to a newer one (the utility function can be specified to capture the probability of an accident and the possibility of loss of utility due to injury in an accident, but in the interest of space we do not bother to go into those details here). When we substitute the demand for driving function $x(a, p_f, \tau)$ back into the utility function, we obtain an *indirect utility* for owning a car of age a that reflects the individual's optimal choice over driving, which we denote as $u_\tau(a)$ to draw the link to our previous notation:

$$u_\tau(a) = u_\tau(a, x(a, p_f, \tau)) - \mu_\tau x(a, p_f, \tau) p_f. \tag{126}$$

If p_f is time-invariant, then our previous analysis already subsumes the case where driving is a static subproblem since we can see that the utility function $u\tau(a)$ can be interpreted as an indirect utility that incorporates an optimal choice of driving within each period. Though actual driving in any period is likely to be relatively variable, we can treat $x(a, p_f, \tau)$ as *planned driving* and consider extensions where *actual driving* responds to *needs for driving* that cannot be predicted exactly *ex ante* by individuals at the start of each period. Let η represent a random variable that captures variable needs for driving that are revealed over time *within each period* and let $u_\tau(a, x, \eta)$ be the utility function of an individual that reflects the *ex post* realization for the needs for driving within a given period. Then we can also allow the individual to take the information contained in η into account in determining how much to drive *ex post* in any given period. Then we have

$$x(a, p_f, \eta, \tau) = \underset{x}{argmax}\, u_\tau(a, x, \eta) - \mu_\tau x p_f \tag{127}$$

so actual realized driving $x(a, p_f, \eta, \tau)$ is a random variable that depends on the need for driving shock η. However if we assume that the consumer makes decisions about *keeping or trading a car* at the *start* of each period *before* the individual observes the need for driving η, then we can simply define $u(a, \tau)$ to be the expected utility that the consumer obtains knowing that they will do their driving optimally within the coming period in response to the information they observe on η that determines their need for driving. So we have

$$u_\tau(a) = E\left\{u_\tau(a, x(a, p_f, \eta), \eta) - \mu_\tau x(a, p_f, \eta, \tau) p_f\right\}. \tag{128}$$

Thus, the model we introduced in the previous sections can be consistent with a range of flexible stochastic models for automobile driving as well as incorporating stochastic shocks that create flexibility to represent variability in the patterns of the discrete holding and trading decisions individuals make as well.

Consider the following specification,

$$u(a,x) = u(a) + \alpha_1(a) + (\alpha_2(a) + \eta)x + \alpha_3(a)x^2 \tag{129}$$

where we assume that η is a normally distributed random variable that represents varying needs for driving that are captured in the linear term of the utility function. We assume that $\alpha_3(a) < 0$ so that there is diminishing marginal utility from driving. Solving for the optimal level of driving $x(a, p_f, \eta)$ implied by this utility function we have

$$x(a, p_f, \eta) = \left(\mu p_f - \alpha_2(a) - \eta\right)/2\alpha_3(a). \tag{130}$$

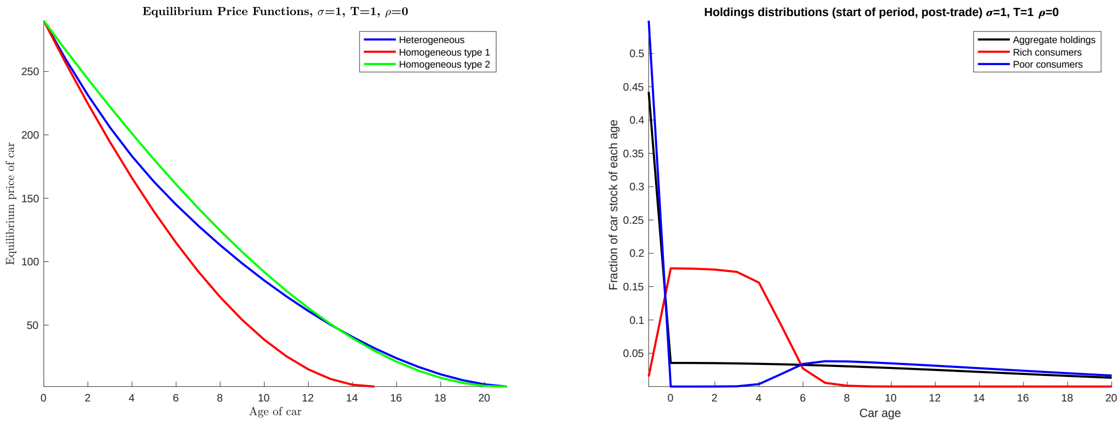
so that optimal driving is represented by a simple linear regression formula.[11]

The value of including driving in the model is that it can enable us to use the model to consider the impact of policy changes that may affect both the price per kilometer of driving p_f and the prices of new and used cars. Consider a specific policy of interest in Denmark, which currently has one of the world's highest tax rates on the purchase of new cars (180%!, from which the Danish government earns approximately 7% of its tax revenues). A policy under consideration is to reduce the new car tax and replace it with either a fuel tax or road user charges (i.e. a tax based on kilometers travelled, x) that would be closely akin to a fuel tax. By incorprating the driving decision into the model we can model this tradeoff.

Figure 6 illustrates the equilibrium prices and holdings under a 180% tax on new cars. The price of a new car is $\overline{P} = 290$ in this equilibrium (which for concreteness we interpret as 290,000 Danish kroner), and this equals the manufacturer retail price of 161,111 plus the 180% tax on new cars. The market reacts to the high price of new cars due to the car tax by also raising the prices of used cars. Consumers also respond by keeping their cars longer in aggregate, so cars

[11] Here we ignore the complication induced by the restriction that $x(a, p_f, \eta) \geq 0$ which implies that η must actually be a truncated normal distribution.

Figure 6: Equilibrium with a 180% new car tax



are not scrapped until they are 21 years old, as you can see in the left panel of figure 6. The right panel illustrates the equilibrium holdings of cars by rich and poor consumers, respectively. Poor consumers are hurt by the high car tax, and 55% of them conclude they cannot afford to own a new or used car and use alternative forms of transportation (e.g. public transportation). The 45% of poor consumers who do own cars own only cars that are 4 years or older. Rich consumers, on the other hand have no problem affording the high price of a new car. Only 1.5% or rich consumers do not own a car, and the remaining 98.5% own cars that are under 8 years old.

In aggregate, the high new car tax discourages new car sales and only 3.55% of the population buys new cars each period, so revenue from the new car tax amounts to 4,575 kroner per capita, or approximately 25.6 billion kroner from the 5.6 million Danish citizens, which is approximately the annual revenue Denmark actually earns from its new car "registration tax."

Now consider the effect of removing the registration tax but replacing it with a fuel tax (or "road user charge" based on kilometers travelled per year) that earns approximately the same annual tax revenue as the registration tax. We used a calibrated version of the quadratic utility function for driving given in equation (129) above to predict the impact of this change in taxation. Prior to the imposition of a fuel tax, the average price per kilometer travelled given the average fuel economy of cars and gas prices in Denmark is $p_f = 0.559$. Using an estimated linear demand for driving given in equation (130) and matching its coefficients to those implied by regression equation $x = 23.5 - 10.9p_f$ estimated using Danish register data, the model implies that prior to the increase in the car tax the average car in Denmark is driven 17,000 kilometers per year.

Figure 7: Equilibrium with no new car tax but a 30% increase in fuel tax

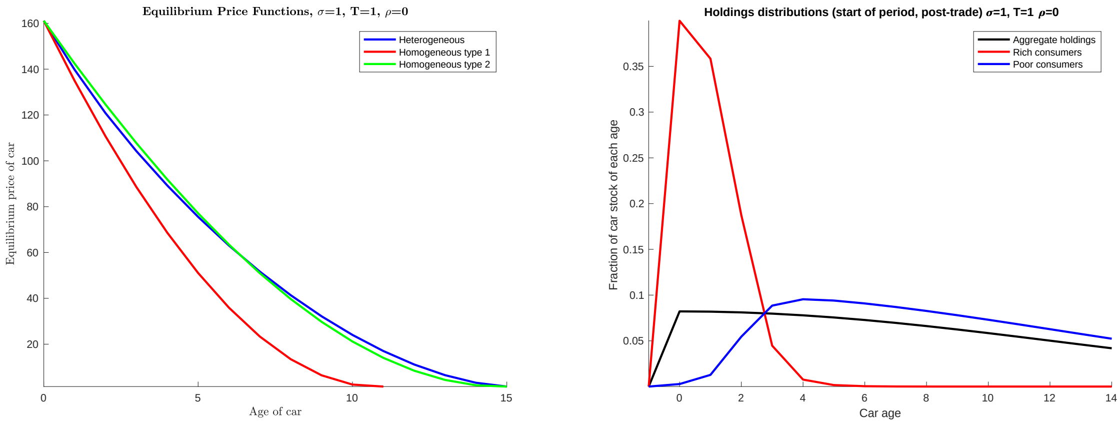


Figure 7 illustrates the equilibrium in the auto market if the new car tax is scrapped and replaced by a 30% tax on the price of gas. The huge reduction in prices of new cars reduces the incentive to keep cars longer to economize on the high cost of purchasing new cars: now 8.2% of Danish households purchase new cars each year, and the scrappage age falls from 21 years to 15. The elimination of the new car tax also induces all consumers in the economy to own cars, though the gas tax induces each car owner to drive 13% fewer kilometers per year. However aggregate driving increases by nearly 60% because all Danish households own cars after the new car tax is eliminated. This is also why a relatively small 30% tax on fuel generates the same revenue as a 180% tax on new cars.

We can calculate consumer expected utility under both tax regimes. Utility of both poor and rich consumers is higher under the fuel tax regime. Valued in kroner, the switch to a 30% fuel tax is worth 162,348 kroner to a rich household and 23,190 kroner to a poor household. In aggregate, the change in tax regime generates a welfare gain of approximately 136 billion kroner for the 2.66 million Danish households. Note, however, that we have not attempted to value the potential loss in welfare due to increase road congestion and travel times that such a huge shift in car ownership might cause.

5 Endogenous Determination of New Car Prices

Now we extend our model to allow for endogenous determination of new car prices in the primary market. We consider the Bertrand-Nash equilibrium definition of equilibrium, but where new car producers take into account not only the effect of price changes on substitution to other new cars and to the outside good, but also its effect on equilibrium in the used car market and on the endogenous deterimination of scrappage ages. Thus, we assume new car producers correctly predict the effect of their collective action in setting new car prices on the overall equilibrium in the car market, both in the primary and in the secondary markets.

We allow for multi-product firms and define a Bertrand-Nash equilibrium where a firm that produces multiple cars internalizes the effect changing prices of its own car prices not just on its competitor market shares, but also on the composition of its own sales of the different types of cars it produces. Following the notation in section 3.6, we let $j \in \{1,\ldots,J\}$ index the set of all cars that could potentially be produced by a total of F firms which we index as $f \in \{1,\ldots,F\}$. Let the sets $(O_1,O_2,\ldots,O_F)$ be a partition of the set of possible car types $\{1,\ldots,J\}$, so that O_f denotes the set of car types owned and potentially produced by firm f. We can think of O_f as a collection of patents and production technologies that firm f has for producing its various car types. We use the qualifier "potentially" since whether or not firm f will actually produce all of the car types that it owns and could produce will be determined as part of the equilibrium in the primary market.

Let $c_f(\vec{q}_f)$ be firm f's cost function for producing the set of cars that it owns, where $\vec{q}_f = \{q_j | j \in O_f\}$ is the vector of quantities of the various cars that firm f produces, normalized as shares of the population that buy new cars of these various types in a steady state equilibrium. Thus, at this level of generality we allow very flexible patterns of production interdependencies, such as economies of scale and economies arising from joint production due to central production of shared underlying vehicle components (e.g. engines or chassis, etc). At this stage of our analysis we ignore the higher level product R&D investments that firms use to introduce new car types into the market. We take the set of cars (and their characteristics) produced by firm f, i.e. the vehicle types in the set O_f, as given, and assume there is common knowledge by all car producers how different types of consumers perceive different types of cars. That is, we assume that there is common knowledge of consumer preferences $u_\tau(a, j)$ and consumer marginal utilities of money, μ_τ by the firms in this market.

Let $\overline{P}_f = \{\overline{P}_j | j \in O_f\}$ be the vector of new car prices that firm f could potentially charge for its own car types $j \in O_f$ in a stationary equilibrium, and let $\overline{P}_{-f}$ denote the new car prices of all other types of cars that are owned and charged by firm f's competitors. In a Bertrand-Nash equilibrium firm f's prices will be set as a best response to the prices charged by its competitors, i.e. we will have

$$\overline{P}_f = \Psi_f(\overline{P}_{-f}), \tag{131}$$

where the function Ψ_f is given by

$$\Psi_f(\overline{P}_{-f}) = \underset{\overline{P}_f}{argmax} \sum_{j \in O_f} \overline{P}_j q(\overline{a}_j(\overline{P}_f, \overline{P}_{-f}), j, \overline{P}_f, \overline{P}_{-f}) - c_f(\{q(\overline{a}_j(\overline{P}_f, \overline{P}_{-f}), j, \overline{P}_f, \overline{P}_{-f}) | j \in O_f\}). \tag{132}$$

Basically, equation (132) states that firm f chooses the prices of its car types, $\overline{P}_f$ to maximize the difference between revenues less cost of production, taking the prices of its competitors' $\overline{P}_{-f}$ as given, but considering the effect of its prices (and those of its competitors) on the overall equilibrium in the new and used markets for cars, including potentially "internal competition" among its own different car types that it could potentialy produce. Though equation (132) provides only single period profits, in a stationary equilibrium if firm f discounts future profits with discount factor $\beta_f \in (0,1)$, then discounted profits equal the right hand side of (132) scaled up by $1/(1-\beta_f)$ which does not affect the definition of Ψ_f.

In equation (132) $q(\overline{a}_j(\overline{P}_f, \overline{P}_{-f}), j, \overline{P}_f, \overline{P}_{-f})$ represents the new car sales of car type $j \in O_f$ in a stationary equilibrium of the model, where q is the invariant distribution of cars, and $\overline{a}_j(\overline{P}_f, \overline{P}_{-f})$ is the scrappage age in the maximal equilibrium when the vector of new car prices set by firm f are given by $\overline{P}_f$ and the vector of new car prices set by its competitors if $\overline{P}_{-f}$. To keep the notation simpler, we have omitted the dependence of all of the used car prices as arguments of the scrappage age function $\overline{a}_j$ and the invariant distribution $q(P)$ where

$$q(P) = \sum_{\tau} q_\tau(P) \Delta_\tau(P) \Omega \tag{133}$$

where $q_\tau(P)$ is the stationary equilibrium holdings distribution for consumers of type τ given in equation (120) above. The prices of all used cars will be implicit functions of the set of new car prices $(\overline{P}_f, \overline{P}_{-f})$ as described in section 3.6. Thus, though each firm takes its competitor prices as given, each firm is presumed to have a sophisticated understanding of the equilibrium in the overall car market, and how the vector of all prices $(\overline{P}_f, \overline{P}_{-f})$ affects sales of all types of new and used cars, and the vector of scrappage ages $\overline{a}_j(\overline{P}_f, \overline{P}_{-f})$, and the stationary holdings distribution $q(\overline{P}_f, \overline{P}_{-f})$ and via this, the effect of prices on the share of all consumers who choose not to own any car.

A Bertrand-Nash equilibrium is then any fixed point $(\overline{P}_1^*, \ldots, \overline{P}_F^*)$ to the system of best response correspondences for the F firms operating in this market:

$$(\overline{P}_1^*, \ldots, \overline{P}_F^*) \in (\Psi_1(\overline{P}_{-1}^*), \ldots, \Psi_F(\overline{P}_{-F}^*)) \tag{134}$$

We now present a simple, stylized analysis of a market with $J = 2$ types of cars and $F = 2$ firms to provide a concrete illustration of the rather abstract equations given above. Figure 8 illustrates the demand curve for new cars and how scrappage and the share of consumers choosing the outside good changes as firm 1 changes the price of its car, $\overline{P}_1$, while firm 2 keeps its car price fixed at $\overline{P}_2 = 220$. The first panel of figure 8 plots $q(\overline{a}_1(\overline{P}_1, \overline{P}_2), 1)$ (new car 1 demand function) and $q(\overline{a}_2(\overline{P}_1, \overline{P}_2), 2)$ (new car 2 demand function) as a function of $\overline{P}_1$. We see, as expected, that new car 1 demand is a downward sloping function of its own price, whereas the demand new cars of type 2 is an increasing function of the price of new cars of type 1, $\overline{P}_1$, reflecting a natural market level substitution effect. Notice a price of $\overline{P}_2 = 220$ for car type 2 appears to cause consumers to regard it as an "overpriced good" in this market in the sense that for sufficiently low values of $\overline{P}_1$ demand for car type 2 falls to zero. Demand for car type 2 only arises when $\overline{P}_1 \geq 180$. When $\overline{P}_1$ rises to 220 (and thus equal to $\overline{P}_2$), demand for car type 1 falls to 0, and the demand for car type 2 peaks at a little over 3% of the population buying new type 2 cars each period in equilibrium.

The middle and lower panels of figure 8 provide more insight into the complexity of market demand when all relevant avenues of consumer substitution are considered. The middle panel show that as $\overline{P}_1$ rises, the secondary market adjusts and the scrappage age $\overline{a}_1(\overline{P}_1, \overline{P}_2)$ increases from 14 to 18. The increase in the scrappage ages reduces new car demand and contributes to the price elasticity of demand for new cars of type 1. The bottom panel of figure 8 shows an additional margin of substitution: an increasing share of consumers choosing not to own cars as $\overline{P}_1$ rises. At sufficiently low prices for $\overline{P}_1$ virtually all consumers own cars but as $\overline{P}_1$ approaches the price of car 2, $\overline{P}_2 = 220$, nearly 60% of the population choose not to own cars. Essentially all of these are in the 80% of the population who are "poor" (i.e. have the higher marginal utility of $\mu_\tau = 2$ compared to $\mu_\tau = 1$ for the "rich" consumers).

Figure 9 illustrates the existence of a unique duopoly Bertrand-Nash equilibrium in this model. We assume that the two firms have constant marginal costs of production given by $c_1 = 150$ for car type 1 and $c_2 = 170$ for car type 2. In this figure we plot the two best response functions Ψ_1 and Ψ_2 as a function of the competitor's price. The reaction functions cross at the pair $(\overline{P}_1^*, \overline{P}_2^*) = (151.84, 186.27)$. In this equilibrium ccar type 1 is actually the "inferior good" in the sense that even though it has a lower marginal cost of production and a lower price, both consumer types prefer car type 2 at the higher price of $\overline{P}_2^* = 186.27$. The markup for car type 1 is only 1.84 whereas the markup for car type 2 is 16.27: evidently, there is something about the utility provided by car type 2 that makes it relatively more valuable to consumers.

Figure 8: Primary market demand as a function of $\overline{P}_1$

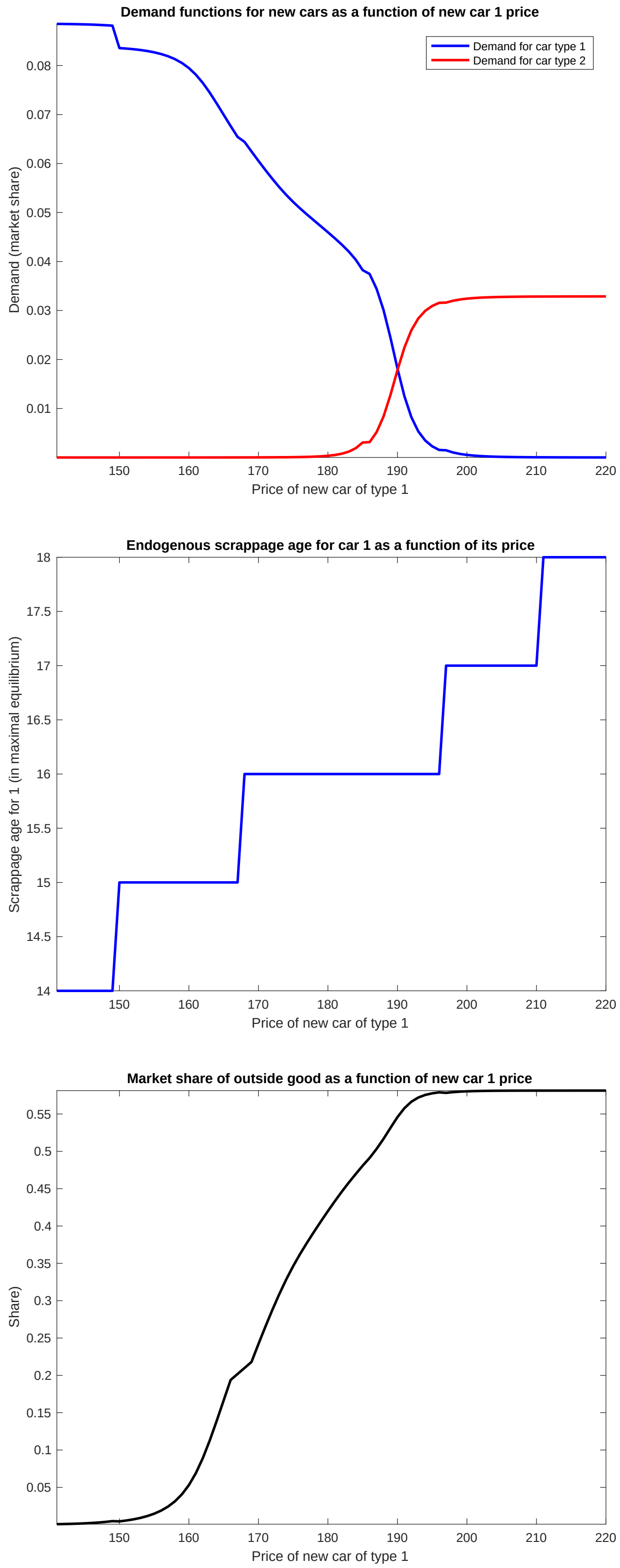


Figure 9: Betrand-Nash Equilibrium in the new car market

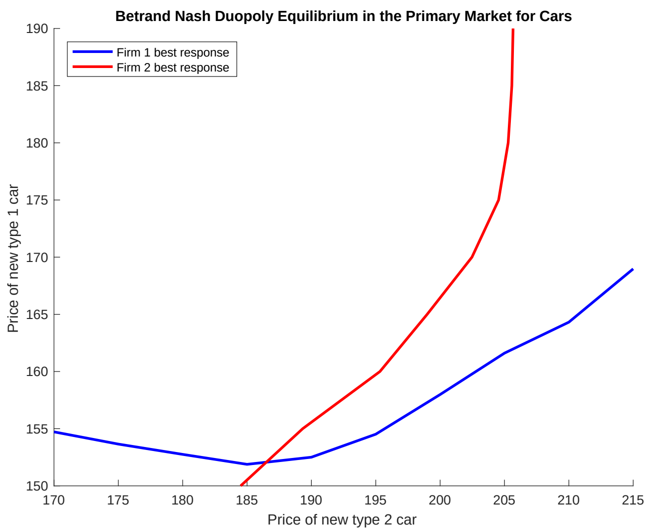


Table 1 provides a final illustration of the usefulness of our equilibrium analysis for policy analysis. Suppose firms 1 and 2 were to merge: what would be the impact of this "merger to monopoly" be on prices, profits and consumer surplus? We can see from the table that allowing such a merger would have devastating impact on the car market: the prices of cars 1 and 2 more than double. The monopolist endogenously "overprices" car type 1 causing its market share to falls nearly to zero. In response to the high prices charged by the monopolist, nearly 78% of the population chooses to forgo owning a car after the merger. Though firm profits triple after the merger, consumer surplus is nearly cut in half and total surplus (consumer surplus plus total profits) falls by 44%. Since consumer surplus is vastly greater than firm profits, it seems that allowing a merger would be a very bad idea in this stylized economy.

There is some similarity in the adverse welfare consequences of the extremely high 180% registration tax imposed by the Danish government and the 107% "tax" a monopolist imposes on sales of its type 2 cars: in both cases the prices of cars have been driven up so high that a huge fraction of the population is unable to even afford them. In the case of a merger, the welfare cost of the prohibitively high new car prices charged by the monopolist falls mainly on poor consumers, who are are nearly entirely driven out of the car market. This is also true in our stylized analysis of the Danish registration tax: its incidence is largely borne by poor consumers. Our very simplified, stylized illustrations should not be taken as serious conclusions about mergers or Danish tax policy, but we believe that more realistically calibrated versions of our model could prove useful for a variety of types of policy analyses.

Table 1: Welfare Analysis of a Hypothetical Merger to Monopoly

Outcome	Monopoly	Duopoly
Price of car 1	374.91	151.84
Price of car 2	386.93	186.27
Profits car 1	3×10^{-4}	0.0165
Profits car 2	3.35	1.09
Total profits	3.35	1.11
Consumer surplus	39.24	74.48
Social surplus	42.59	75.59
Consumer surplus rich, cars 1/2	0.002/32.54	2.66/26.33
Consumer surplus poor, cars 1/2	0.0004/6.69	3.23/42.26
Fraction buying new cars 1/2	1.5×10^{-4}%/1.5%	0.9%/6.7%
Fraction owning no car	77.9%	0.3%
Fraction of rich owning no car	2.4%	0.0%
Fraction of poor owning no car	96.8%	0.04%
Car 1/2 market shares, rich	7×10^{-3}%/97.6%	11.5%/88.5%
Car 1/2 market shares, poor	0%/3.2%	10.5%/89.1%
Scrappage ages cars 1/2	18/20	15/18
Average age held, cars 1/2, poor	17.15/18.27	8.8/9.9
Average age held, cars 1/2, rich	7.80/7.85	2.00/2.15

6 Conclusion

We have introduced a computationally tractable model of equilibrium in the primary and secondary markets for automobiles that allows for flexible specifications of preferences and consumer heterogeneity and transactions costs. Our work was inspired by the early static, discrete choice models of equilibrium in the automobile market pioneered by Manski, Sherman, and Berkovec and the subsequent efforts to extend their models to include dynamics and transactions costs and model equilibrium price setting in the primary market by Rust, Stolyarov, and Esteban and Shum. We believe that our framework is promising for empirical applications and policy analysis, and in future work we plan to further extend and apply it in a number of directions. First, in ongoing work Gillingham, Iskhakov, Munk-Nielsen, Rust and Schjerning (2019) we are econometrically estimating an "overlapping generations" version of our model using Danish register data that we hope can provide the basis a more "serious" analysis of car tax reform policy in Denmark.

Second, we are developing alternatives to the Berry, Levinsohn and Pakes (1995) approach to estimating consumer preferences and analyzing equilibrium in automobile markets. We noted that their approach is based on a static model that fails to consider the relevant substitution margins available to consumers provided by secondary markets for cars, and the nature of endogenous, self selected patterns by heterogeneous consumers in these markets. Our model can account for these features and in future work we plan to use our model to analyze the identification of structural econometric estimators in the context of a more realistic dynamic model of the auto market. We find severe identification problems associated with using only aggregate market share data and instrumental variable approaches to estimate these models.

Finally, we are working on relaxing the assumption of stationarity and extend our concept of equilibrium to allow for macro shocks that can capture the pronounced "waves" in the age distribution of cars that we observe in the data. We are comparing different solution concepts in terms of computational tractability and empirical realism, including the "temporary equilibrium" concept of Grandmont (1977), the "sufficient statistic" approach of Krusell and Smith (1998), as well as a full blown rational expectations equilibrium that takes into account the entire holdings distribution of cars as a component of the "state variables" that consumers use to predict future prices as in Cao (2016).

A very challenging extension is to use the model to endogenize the attributes and characteristics of vehicles which our model treats as exogenous and pre-determined. The essential part of competition in the new car market is to conduct R&D to produce new car designs and features that consumers value. This longer run competition on "product attributes" will likely require a fundamentally non-stationary framework with complicated issues of how consumers form expectations over future products that are not yet introduced into the market. Very promising headway into this sort of analysis has been done in the pioneering work of Goettler and Gordon (2011) and it may be possible to adapt their approach to endogenize R&D and new product development into a more evolutionary model of the automobile market.

Another challenging direction is to attempt to incorporate asymmetric information and a more detailed treatment of the "microstructure" of trade in the auto market, including endogenous intermediation of trade by car dealers as well as direct consumer to consumer transactions. Recent studies such as Biglaiser, Li, Murry and Zhou (2019) have provided new empirical insights into the microstructure of trade that will be important directions for modeling. They find that most of the transactions for relatively young used cars are intermediated by dealers (and at a higher price, or transactions cost) whereas most of the trade in relatively older cars occurs via direct consumer to consumer transactions at generally lower transactions costs. Car dealerships also play an important role via inventory holdings that help to buffer short term imbalances in supply and demand. These important features of trade in automobile markets have been ignored in our model but represent important directions to pursue in the development of more detailed and realistic models of trade in automobile markets.

Appendix 1: Equilibrium with persistent, time-varying heterogeneity

Sections 3.1 and 3.2 covered the two polar cases: a) all heterogeneity in consumers is of the form of *IID* preference shocks, and b) in addition to the *IID* extreme value heterogeneity there is time invariant heterogeneity in the form of a finite number of fixed consumer types. This appendix covers the intermediate case where there is heterogeneity that is time-varying and persistent. To fix ideas, introduce a Markovian state variable income y that takes two possible values, low income y_l and high income y_h. We assume that income evolves in a serially correlated manner with a transition probabilty $\pi(y'|y)$ and corresponding 2×2 transition probability matrix M

$$M = \begin{bmatrix} \pi(y_l|y_l) & \pi(y_h|y_l) \\ \pi(y_l|y_n) & \pi(y_h|y_h) \end{bmatrix}. \tag{135}$$

Let π also denote the invariant probability distribution for the income process, i.e. π is the unique solution to

$$\pi = \pi M \tag{136}$$

so $\pi(y_l)$ represents the fraction of consumers with low income in the economy in steady state and $\pi(y_h) = 1-\pi(y_l)$ represents the fraction of high income consumers. In a stationary equilibrium these fractions do not vary over time as we have not allowed for any macro shocks that could could cause correlation across consumers in their idiosyncratic indpendently evolving income processes.

We assume that the level of income could affect car choices in two different ways: 1) y could enter the utility of owning a car $u(a,y)$, or 2) income could enter the marginal utility of money $\mu(y)$. For example if $\mu(y)$ is lower when y is high, this can generate an idiosyncratic motivation for either purchasing a car (for consumers who do not hold a car) or trading an existing car for a newer one.

Though we do not repeat the Bellman equations to conserve on space, let $v(d,a,y)$ denote the value function for a consumer who makes choice d in car state a when their income is y. In general, income will affect the values of different choices in different states, and this will induce different choice probabilities. Let $\Pi(d|a,y)$ be the choice probability of a consumer whose car state is a and whose current income is y. Since y affects utilities, and the values $v(d,a,y)$ it will clearly affect the choice probabilities for consumers as well.

Clearly all consumers who currently have high income will behave differently from those who have low income, and this will affect their post-trade holdings of cars. Let $\Delta(y,P)$ be the $\overline{a}+1\times\overline{a}+1$ transition probability matrix for consumers in income state y given the current price vector P. This is the same *post-trade transition probability matrix* given in equation (51) except that we now allow for the effect of income y on this transition probability matrix.

We now claim there will be a stationary equilibrium in this market that takes the following form. Let q be the overall distribution of car holdings for the overall population. As in the case of time-invariant heterogeneity analyzed in the previous section, q must be be time invariant in the stationary equilibrium and statisfy the key condition,

$$q = q\Omega \tag{137}$$

However at any time, just after trading there will be two different holdings distributions for low and high income consumers. Let q_y be given by

$$q_y = q\Delta(y,P)\Omega. \tag{138}$$

Thus, q_y represents the holdings of cars for consumers whose income is currently equal to y just after trade occurs. If $\pi(y)$ is the invariant probability of income equalling level y, then we have

$$q = \sum_y \pi(y)q_y = \sum_y q\Delta(y,P)\Omega\pi(y) = q\Delta(P)\Omega \tag{139}$$

where $\Delta(P)$ is the the averaged post-trade transition probability matrix given by

$$\Delta(P) = \sum_y \pi(y)\Delta(y,P) \tag{140}$$

So we see that in a stationary equilibrium q is also an invariant probability of the average post-trade transition probability matrix $\Delta(P)$ just as we found in the case of an economy with no persistent heterogeneity (see equation (**??**) in Proposition 1). Equilibrium prices will be given by the solution to the same set of excess demand equations as in the case of of the "homogeneous agent equilibrium" (or more specifically the equilibrium with no persistent heteorgeneity amongst consumers). That is P will solve $ED(P) = 0$ where ED is the mapping given in equation (**??**) above, with the only difference being that the choice probabilities entering the definition of equilibrium are averaged by the invariant probabilities $\pi(y)$ of the different possible income states y,

$$\Pi(d|a,P) = \sum_y \Pi(d|a,y,P)\pi(y). \tag{141}$$

We conclude this section by discussing the difference between equation (138) which provides the distribution of holdings of cars at the start of period $t+1$ for consumers who had income y in period t when they executed their trades at the equilibrium prices P versus the type-specific holding distribution q_τ given by the invariant distribution in equation (66) in the previous section. The key difference is that in the case of q_y we use the overall averaged holdings distribution q as the distribution of holdings multiplying the $\Delta(y,P)\Omega$ transition matrix in equation (138) whereas in the case of time-invariant heterogeneity we used q_τ as the holdings distribution multiplying the matrix $\Delta_\tau(P)\Omega$ in equation (66).

The reason for this difference is that when there is time-invariant heterogeneity, a consumer who is type τ in period t will remain a type τ consumer in period $t+1$ with probability 1, and thus, the relevant distribution of holdings for consumers of type τ is q_τ in both periods t and $t+1$, and it follows that q_τ must be an invariant distribution for the transition probability matrix $\Delta_\tau(P)\Omega$. However when there is persistent by not completely time invariant heteorgeneity, the relevant distribution of holdings corresponding to the set of consumers who have income y at time t is the overall average distribution q rather than the income y specific holdings distribution q_y.

To see this, note that at time t the fraction of individuals who have a given income y (say low income y_l) is equal to a weighted average of consumers who previously had high income and those who previously had low income:

$$\pi(y_l) = \pi(y_l)\pi(y_l|y_l) + \pi(y_h)\pi(y_l|y_h) \tag{142}$$

Similarly, the distribution of holdings for all consumers who have low income at time t will be a weighted average of the income-specific holdings distributions q_y given in equation (138) above

$$q = \pi(y_l)q_{y_l} + \pi(y_h)q_{y_h} \tag{143}$$

Thus in a stationary equilibrium the "mixing" between consumers of different incomes generates a stationarity in overall holdings even though we will see in every time period differences in the holdings of cars of rich and poor consumers, both before and after trade occurs.

It is easy to see, then, from the forgoing discussion how to define equilibrium in cases where we have both time varying and time-invariant heterogeneity. For example if τ indexes time-invariant heterogeneity in consumers (i.e. different "types" of consumers) and y indexes time-varying state variables that result in time varying heterogeneity in consumers, then we can combine the equilibrium conditions in this section and the previous section to define a stationary equilibrium as a price vector P that sets excess demand $ED(P)$ to zero, where

$$ED(P) = D(P) - S(P) \tag{144}$$

and the supply and demand functions are given in equations (68 and (69) above respectively, where q_τ is given by the invariant distribution to equation (66) but with the main difference is that the choice probabilities entering equations (68) and (69) are weighted average of the y-specific transition probabilities

$$\Pi(d|a,\tau,P) = \sum_y \Pi(d|a,\tau,y,P)\pi(y) \tag{145}$$

where π is the invariant distribution of the $\{y_t\}$ process (which is again an idiosyncratic process that evolves indpendently over different consumers), and the $\Delta_\tau(P)$ matrix that constitutes the post-trade transition probability matrix is also a weighted average of the y-specific transition matrices

$$\Delta_\tau(P) = \sum_y \Delta_\tau(y,P)\pi(y). \tag{146}$$

Thus, we can define $q_{\tau,y}$ to be the holdings of type τ consumers who had income y in the previous period by

$$q_{\tau,y} = q_\tau D(\tau,y,P)\Omega \tag{147}$$

and clearly we will have

$$q_\tau = \sum_y q_{\tau,y}\pi(y) \tag{148}$$

and finally we have the overall adding up condition of Proposition 3 continuing to hold, i.e.

$$q = \sum_\tau q_\tau f(\tau) = q\Omega. \tag{149}$$

Finally it is possible to further extend this to allow the transition dynamics for the time-varying persistent heterogeneity y to depend on τ. So if $\pi(y'|y,\tau)$ is the transition probability for income for a type τ consumer, then following our previous abuse of notation, then let π_τ also denote the invariant distribution for income. Then it is easy to see all of the equations defining equilibrium hold except with π replaced by π_τ in the equations above.

Appendix 2: Gradient of invariant distribution

In this appendix we consider the general problem of calculating the derivative of an invariant distribution with respect to parameters affecting a Markov transition matrix. Let the parameters be θ (in our application θ is a vector of prices of cars in a secondary market equilibrium) and consider a Markov transition probability matrix $P(\theta)$ that depends on these parameters in a continuously differentiable fashion. Thus, we assume that the mapping $\nabla_\theta P(\theta)$ from R^k to R^{k*n*n} (where the latter can be interpreted as the space of k-tuples of $n \times n$ matrices) exists and is a continuous function of θ. To make things easier to understand, assume initially that $k = 1$ so we are considering $P(\theta)$ and $q(\theta)$ as functions of a single parameter θ. If θ has k components (i.e. $\theta \in R^k$) we simply "stack" the formulas we provide below in the univariate case into a k-tuple.

We are interested in determining the conditions under which $q(\theta)$, the unique invariant distribution of $P(\theta)$, is a continuously differentiable function of θ and, if so, to find an expression for $\nabla_\theta q(\theta)$. The invariant distribution $q(\theta)$ satisfies the equation

$$q(\theta) = q(\theta)P(\theta), \tag{150}$$

which can be recast as $q(\theta)$ being a left zero of the matrix $I - P(\theta)$, $q(\theta)[I - P(\theta)] = 0$. The usual application of the Implicit Function Theorem applies when $q(\theta)$ can be written as a zero of some continuously differentiable nonlinear mapping $F(q, \theta) = 0$ with the added condition that $\nabla_q F(q, \theta)$ is nonsingular at a zero of F. Then the Implicit Function Theorem guarantees that there is a continuously differentiable function $q(\theta)$ in a neighborhod of this zero, and we have

$$\nabla_\theta q(\theta) = -[\nabla_q F(q(\theta), \theta)]^{-1} \nabla_\theta F(q(\theta), \theta). \tag{151}$$

However this usual application of the Implicit Function Theorem is inapplicable because in this case $\nabla_q F(q, \theta) = I - P(\theta)$ and this matrix is singular (note that if e is a vector of ones, then $[I - P(\theta)]e = 0$ where 0 is a vector of zeros). Thus, we have to approach this problem from a different angle.

When the invariant distribution is unique, it can be shown that $q(\theta)'$, the $n \times 1$ transpose of $q(\theta)$, is the unique solution to the expanded $(n+1) \times (n+1)$ linear system given by

$$\begin{bmatrix} I - P(\theta)' & e \\ e' & 1 \end{bmatrix} \begin{bmatrix} q(\theta)' \\ 1 \end{bmatrix} = \begin{bmatrix} e \\ 2 \end{bmatrix} \tag{152}$$

where e is an $n \times 1$ vector all of whose elements equal 1. Thus, the matrix on the right hand side of equation (152) is invertible and we can write

$$\begin{bmatrix} q(\theta)' \\ 1 \end{bmatrix} = \begin{bmatrix} I - P(\theta)' & e \\ e' & 1 \end{bmatrix}^{-1} \begin{bmatrix} e \\ 2 \end{bmatrix}. \tag{153}$$

Let $A(\theta)$ be the $(n+1) \times (n+1)$ matrix on the right hand side of equation (152). Then we have that $\nabla_\theta q(\theta)'$ is the upper left $n \times n$ submatrix of the product of $\nabla_\theta A^{-1}(\theta)$ times the vector $(e'\ 2)'$. Further, we use the following formula for the gradient of $A^{-1}(\theta)$ with respect to θ

$$\nabla_\theta A^{-1}(\theta) = -A^{-1}(\theta)\,[\nabla_\theta A(\theta)]\,A^{-1}(\theta). \tag{154}$$

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